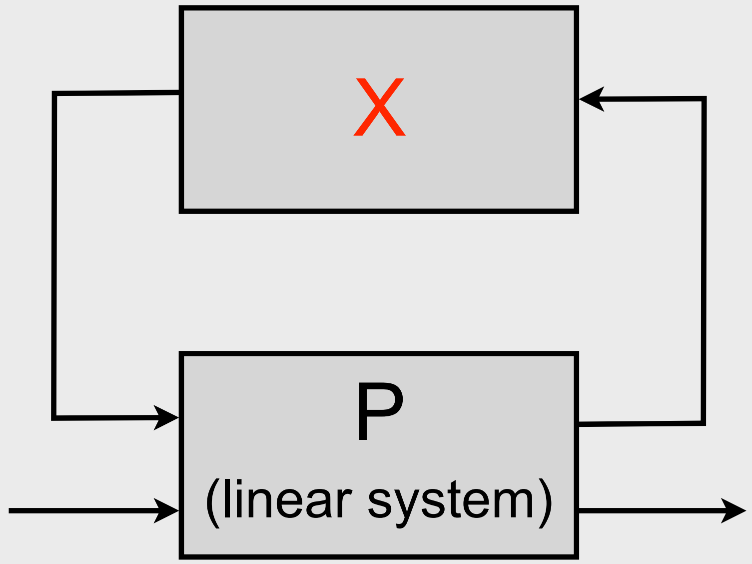
Affine Multiplexing Networks:

System Analysis, Learning, and Computation

Ivan Papusha (joint work with Ufuk Topcu)
Institute for Computational Engineering and Sciences
University of Texas at Austin

X-in-the-loop control

(and verification for it)



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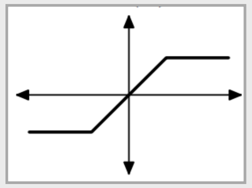
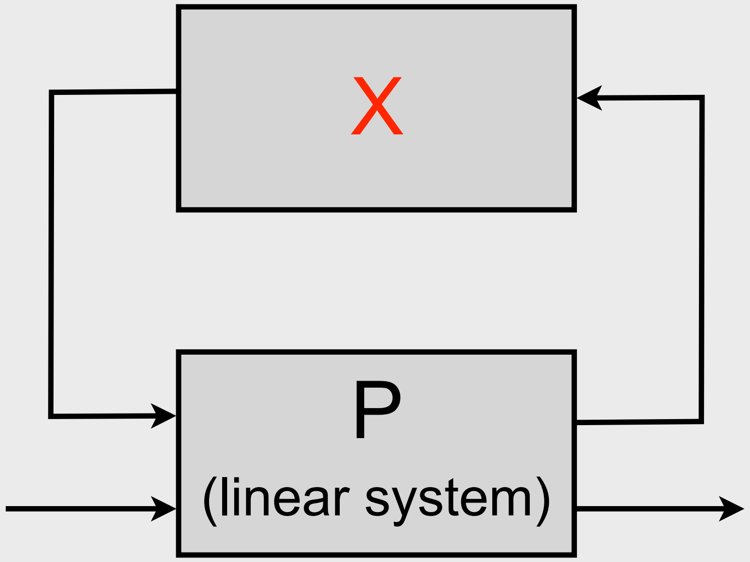
saturation X = 



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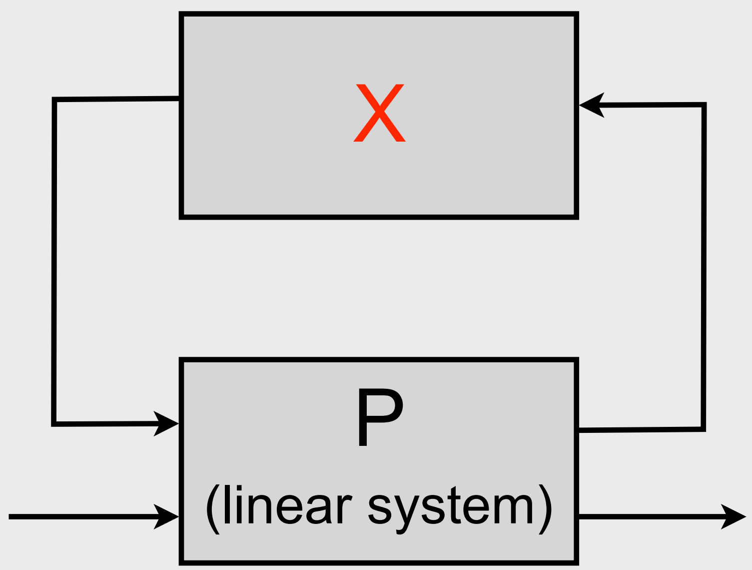
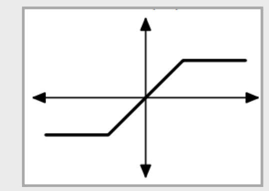


saturation X = 

software X =

```
function [c1, c2, c3] = mscript(gen1, gen2)
global state;
switch (state)
case 0:
  if gen1 == 1 and gen2 == 1 then
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  else if gen1 == 0 and gen2 == 0 then
    state = 1; c1 = 0; c2 = 0; c3 = 0;
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  end if
case 1:
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end switch
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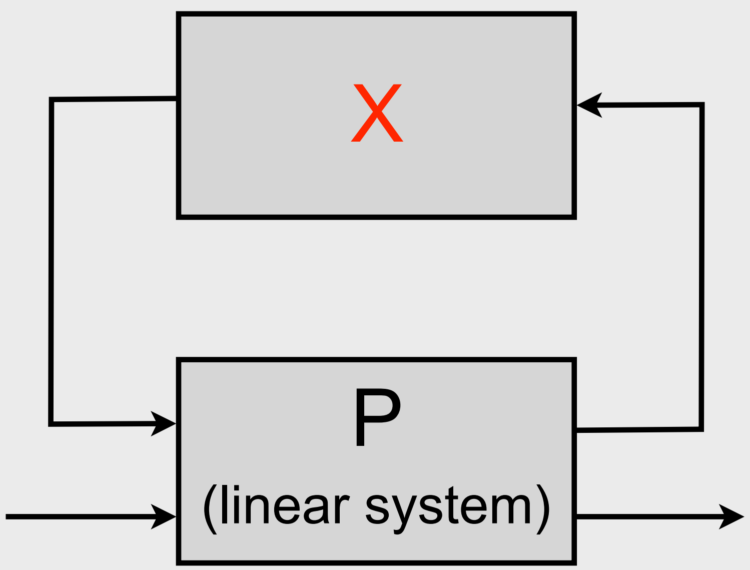
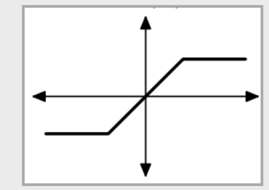
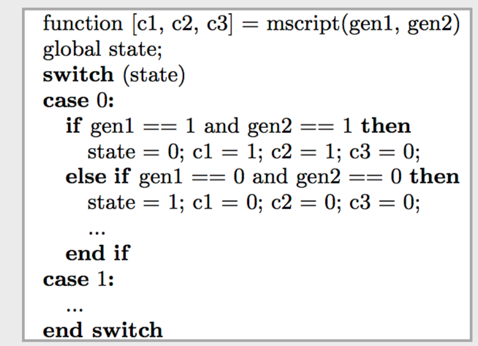


saturation X = 

software X = 

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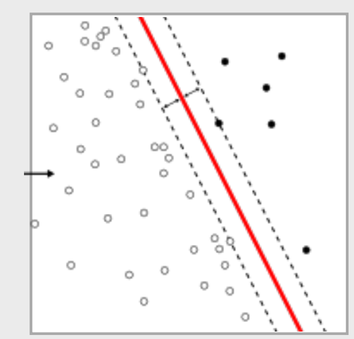
classifier X = 

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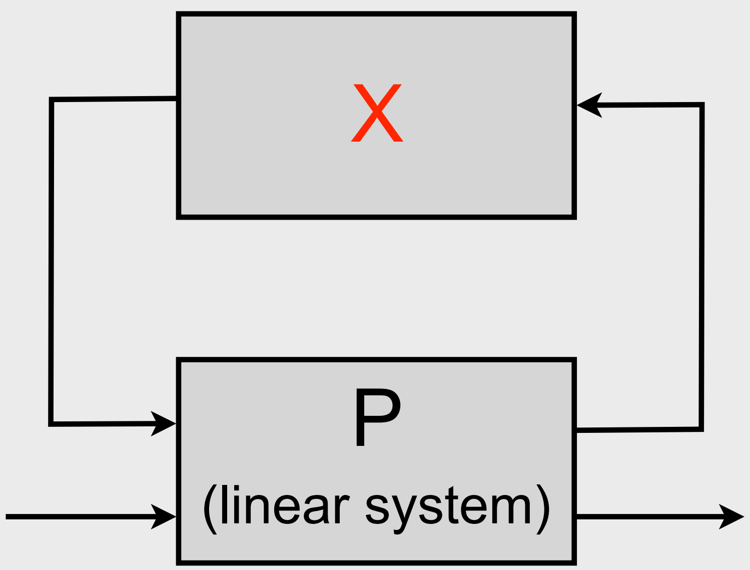

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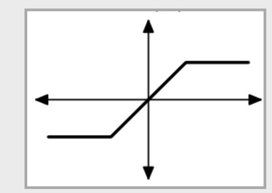

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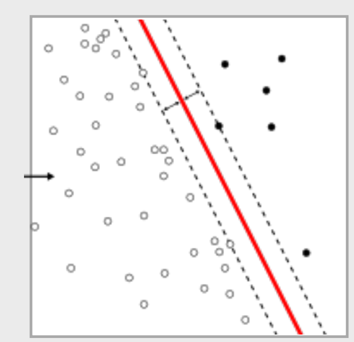
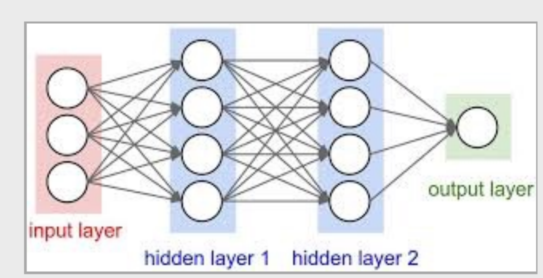


neural network X =



Affine Multiplexing Networks (AMNs)

A directed interconnection of…

1) affine transformations

$$x \longrightarrow \triangleright \longrightarrow y = \alpha(x) = Wx + b$$

2) multiplexing functions

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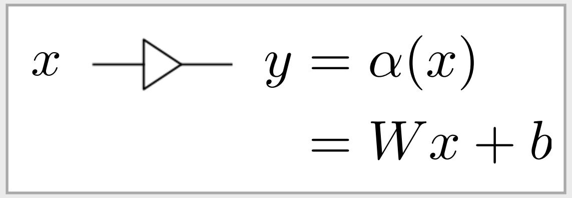
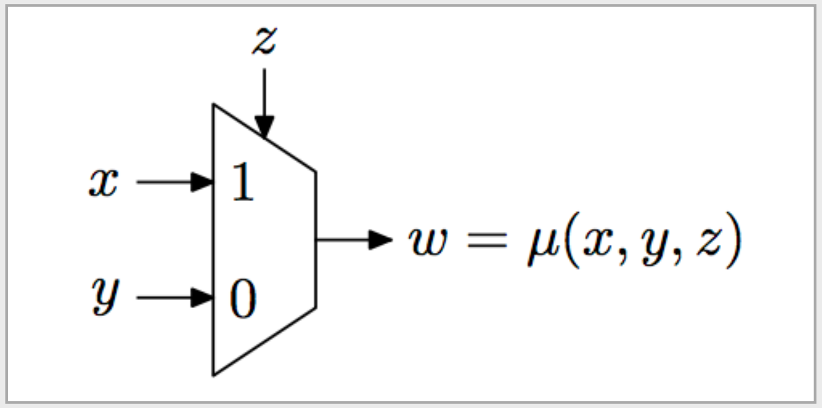


2) multiplexing functions

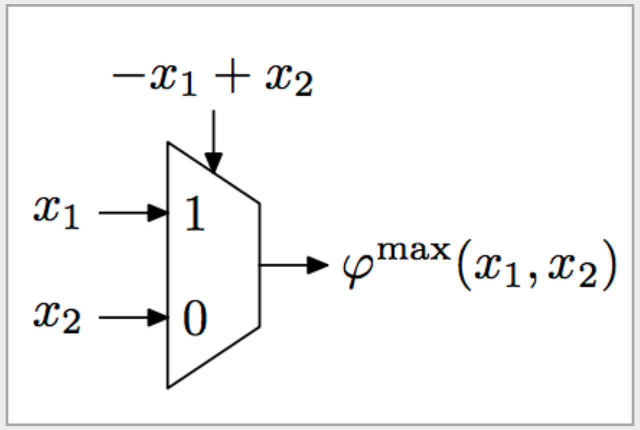


$$\mu : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

$$\mu(x, y, z) = \begin{cases} x, & \text{if } z \leq 0 \\ y, & \text{otherwise} \end{cases}$$

Examples of AMNs

maximum



$$\mu(x_1, x_2, -x_1 + x_2)$$

Examples of AMNs

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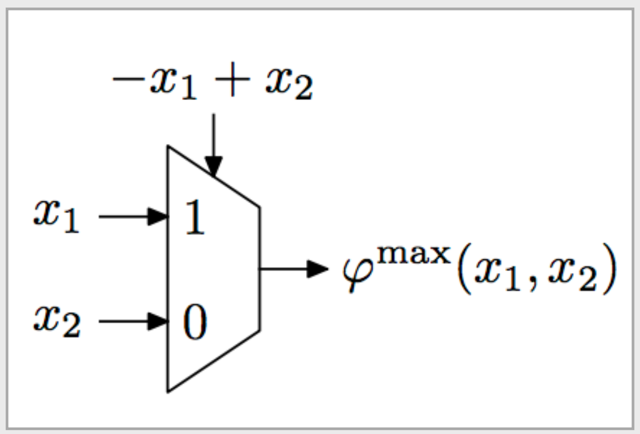
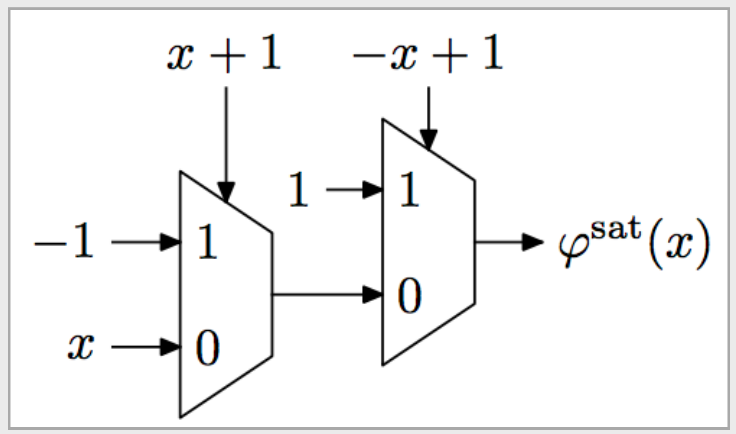


saturation



$$\mu(x_1, x_2, -x_1 + x_2)$$

$$\mu(1, \mu(-1, x, x+1), -x+1)$$

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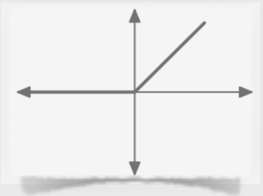

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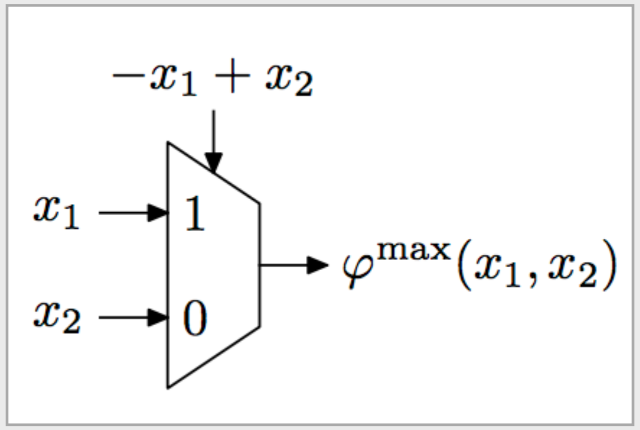

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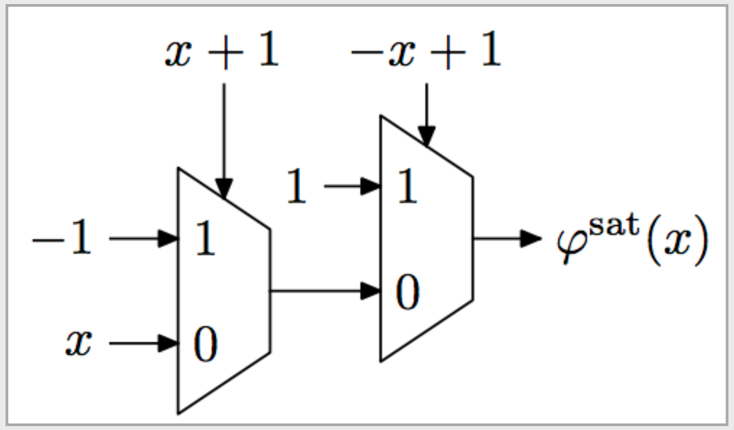
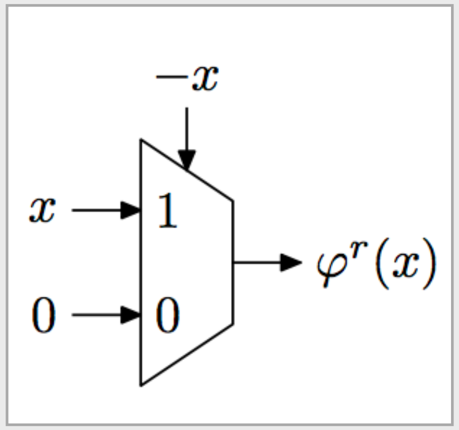
saturation



$$\mu(1, \mu(-1, x, x + 1), -x + 1)$$

ReLU

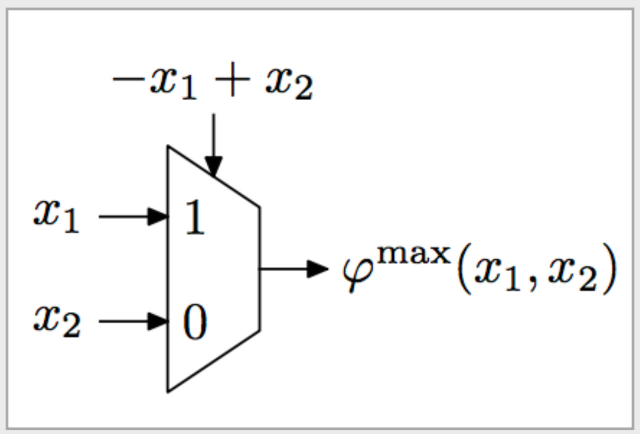
(rectified linear unit)



$$\mu(x, 0, -x)$$

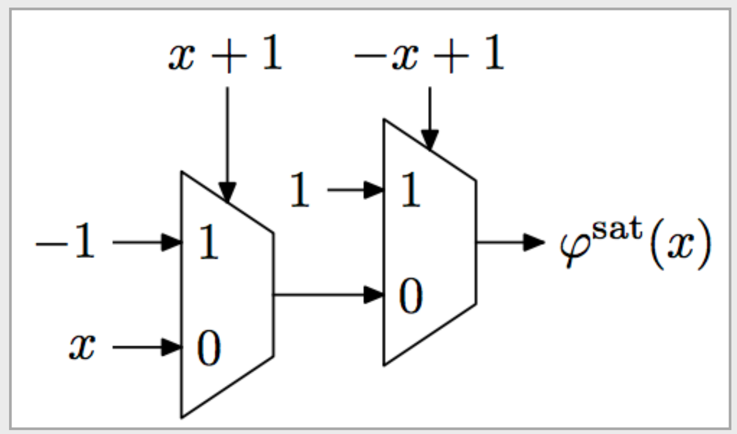
Examples of AMNs

maximum



$\mu(x_1, x_2, -x_1 + x_2)$

saturation



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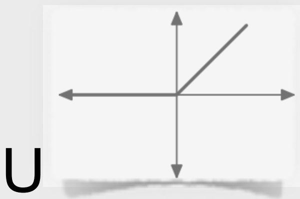

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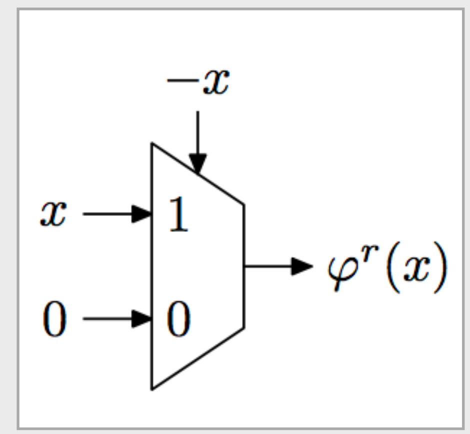
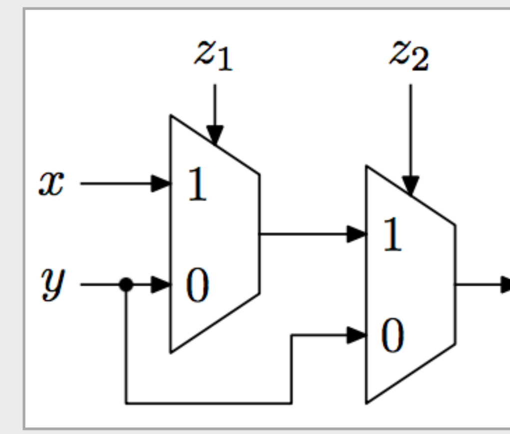


$\mu(x, 0, -x)$

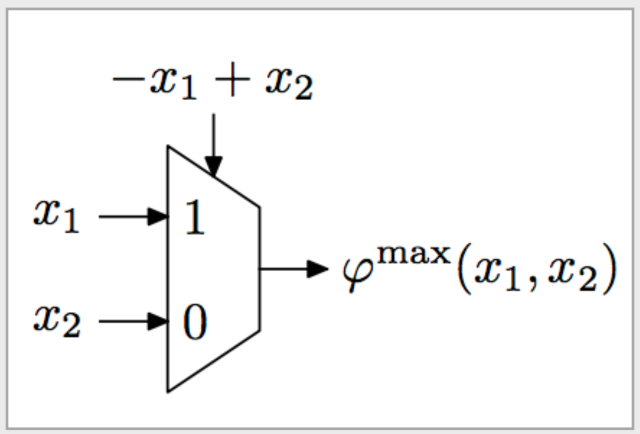
AND gate



$\mu(\mu(x, y, z_1), y, z_2)$

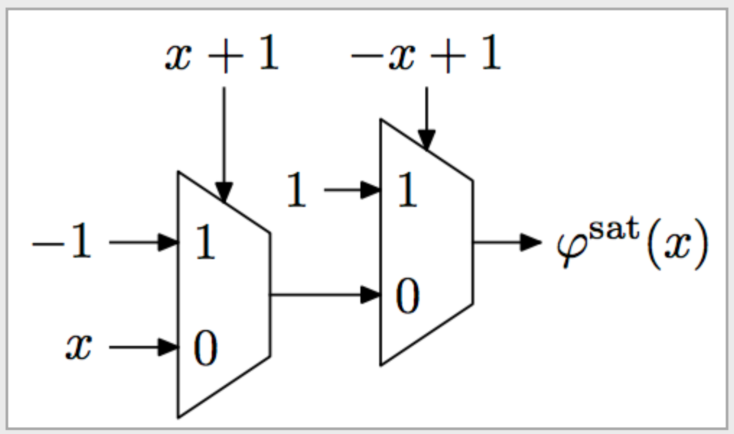
Examples of AMNs

maximum



$\mu(x_1, x_2, -x_1 + x_2)$

saturation



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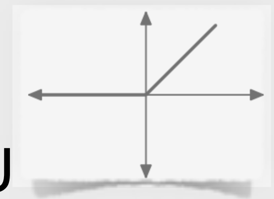

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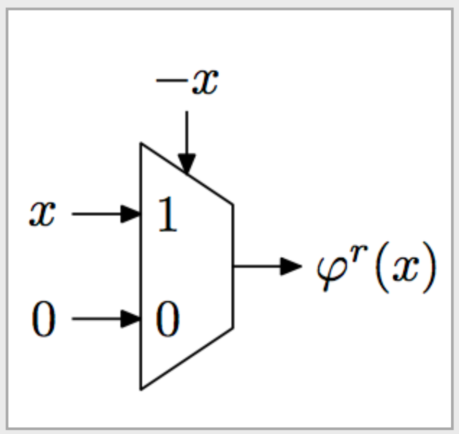

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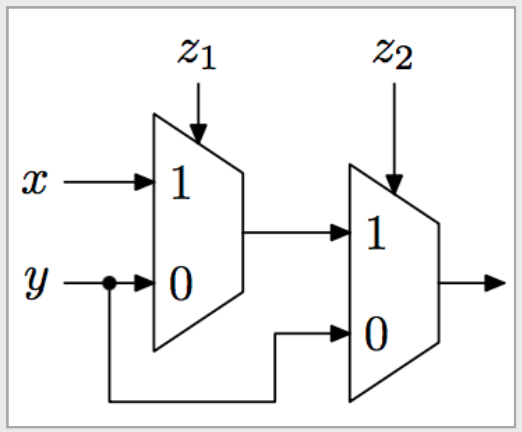
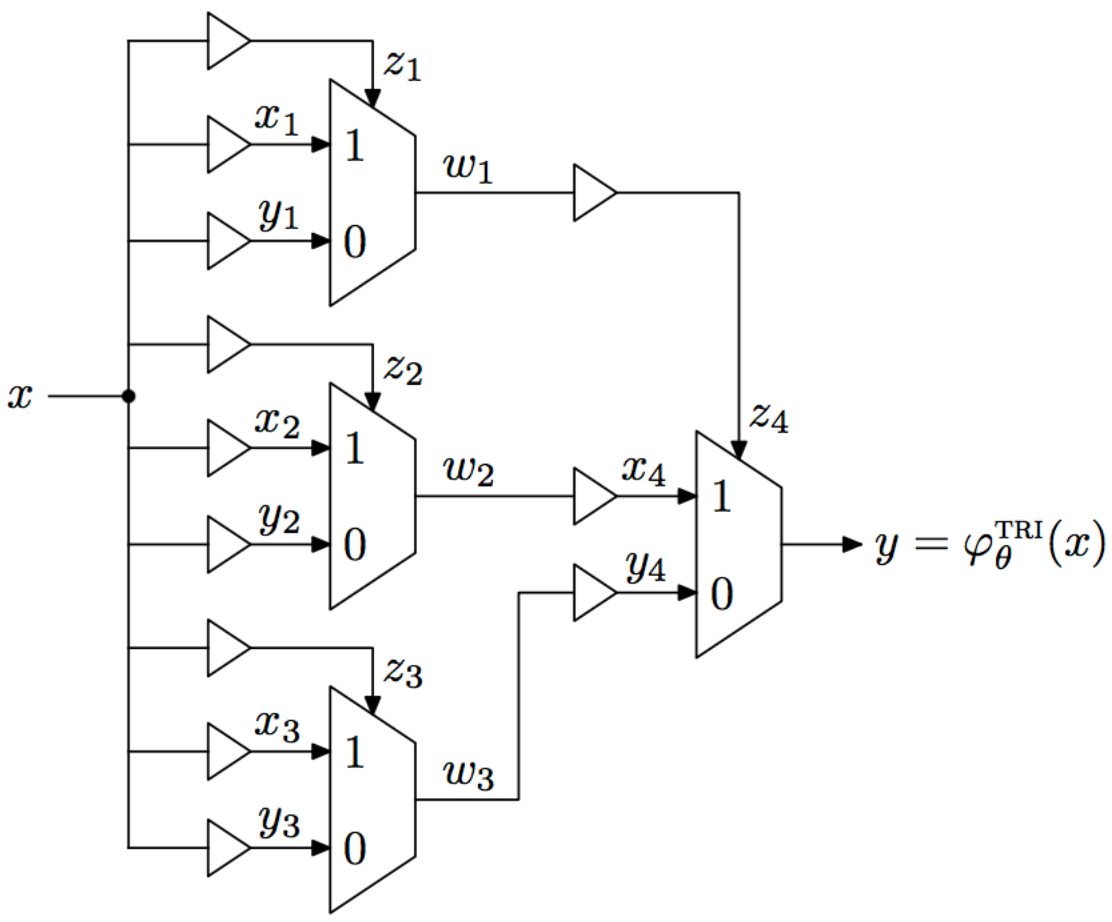


$\mu(\mu(x, y, z_1), y, z_2)$

$\mathbf{card}(x)$	$\sum_{i=1}^{n} \mu(\mu(1, 0, x_i), 0, -x_i)$
$\|x\|_\infty$	$\mu(x_1, \mu(x_2 \ldots), -x_1 + \mu(x_2 \ldots))$
XOR	$\mu(\mu(y, x, z_1), \mu(x, y, z_1), z_2)$
LE	$\mu(x, y, z)$

AMNs compose

A 2-layer,
4-mux
AMN



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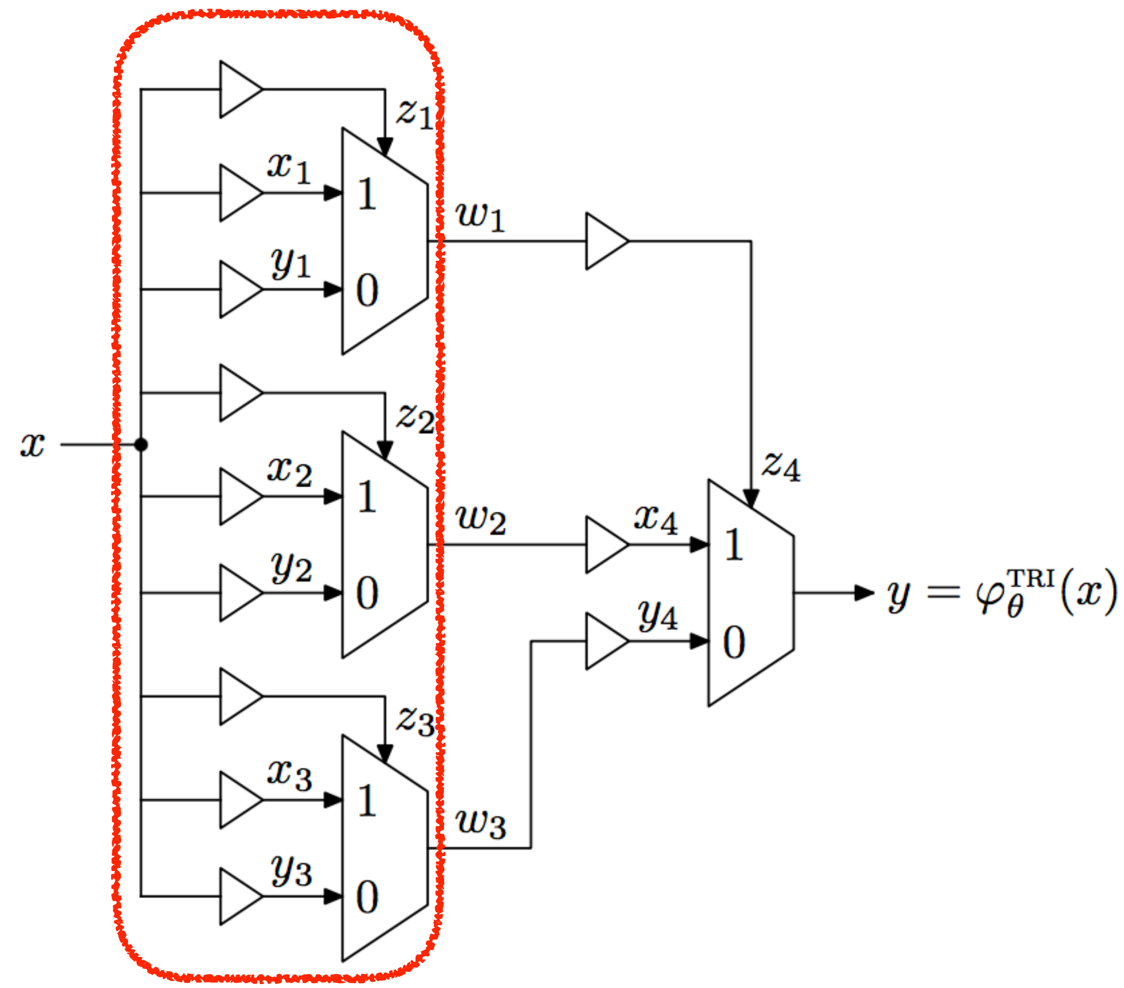

layer 1

weights

$$x_1 := a_1 x + b_1 \quad \cdots \quad x_3 := a_3 x + b_3$$
$$y_1 := c_1 x + d_1 \quad \cdots \quad y_3 := c_3 x + d_3$$
$$z_1 := e_1 x + f_1 \quad \cdots \quad z_3 := e_3 x + f_3$$

nonlinearities

$$w_1 := \mu(x_1, y_1, z_1)$$
$$w_2 := \mu(x_2, y_2, z_2)$$
$$w_3 := \mu(x_3, y_3, z_3)$$

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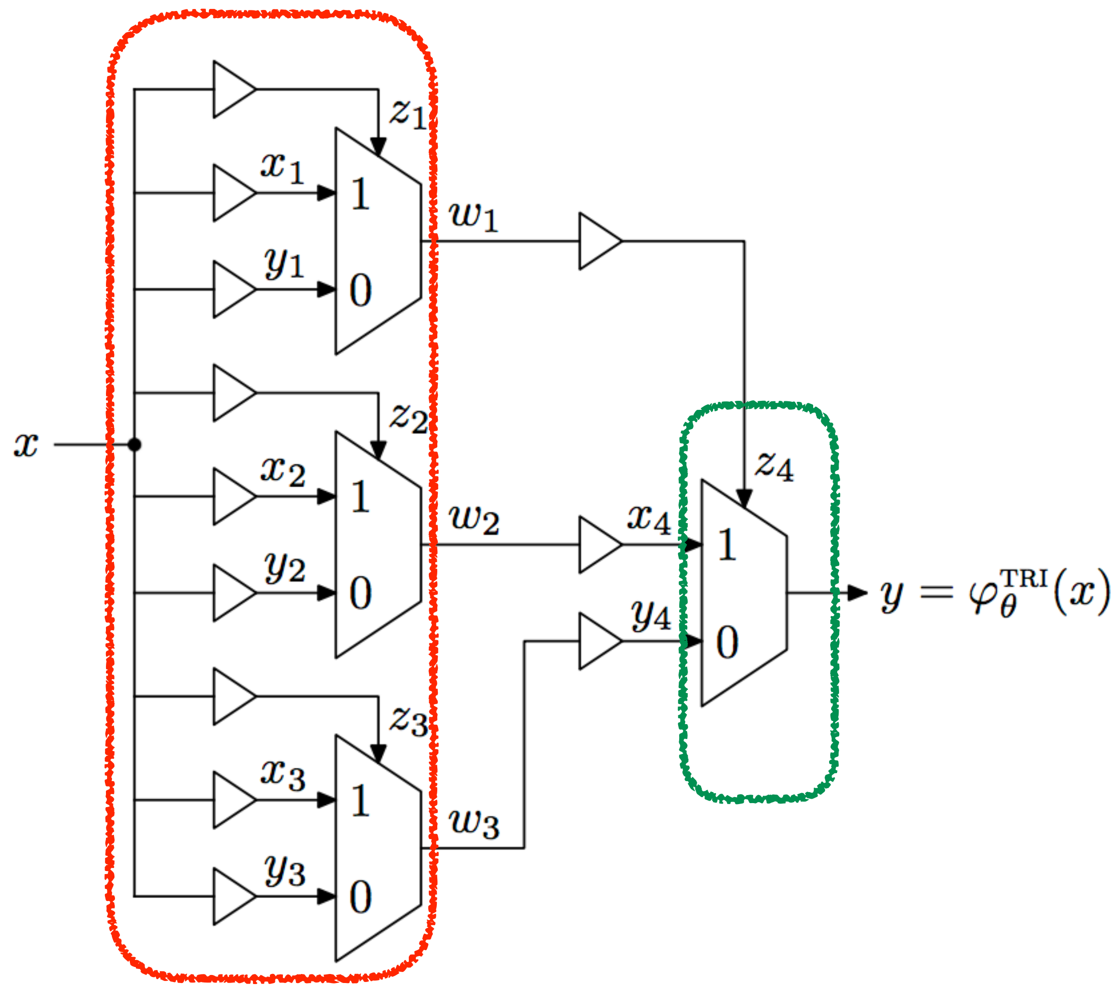

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$$w_3 := \mu(x_3, y_3, z_3)$$

layer 2

weights

$$x_4 := a_4 w_2 + b_4$$
$$y_4 := c_4 w_3 + d_4$$
$$z_4 := e_4 w_1 + f_4$$

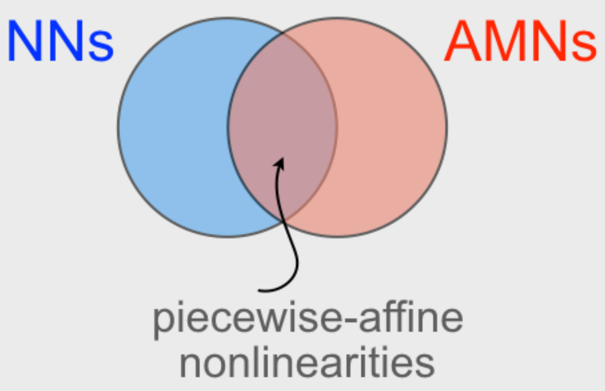
nonlinearity

$$y := \mu(x_4, y_4, z_4)$$

Why care about AMNs?

AMNs are expressive.

Deep multilayer feedforward networks with **piecewise-affine nonlinearities** are a special case.



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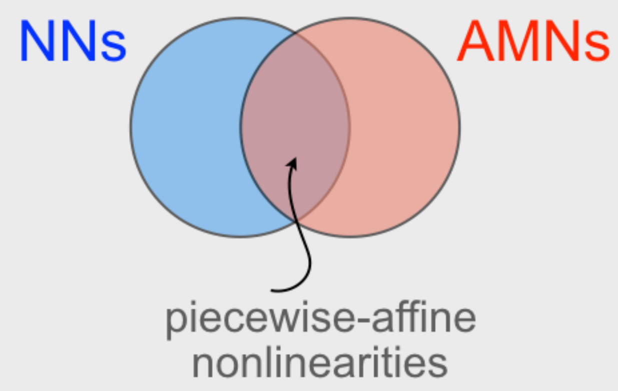
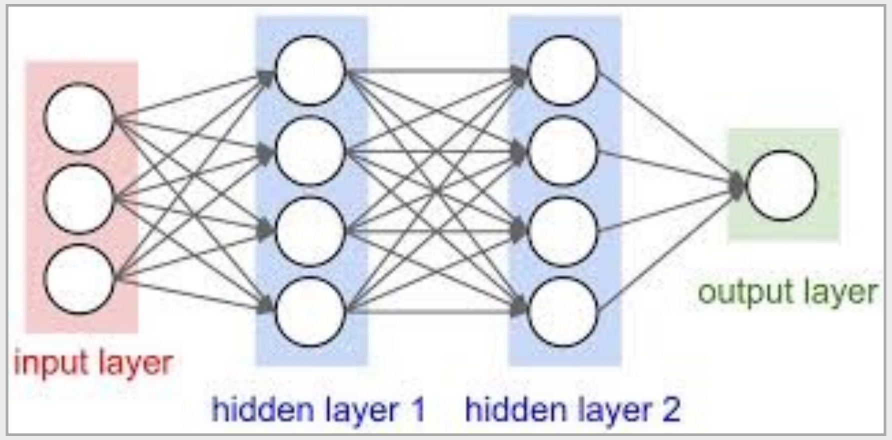
Can automatically translate them into AMNs.





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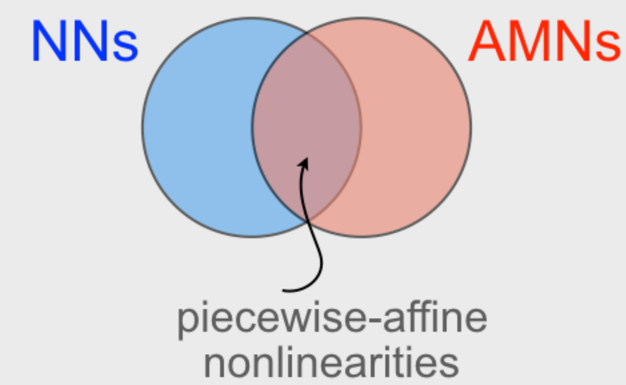

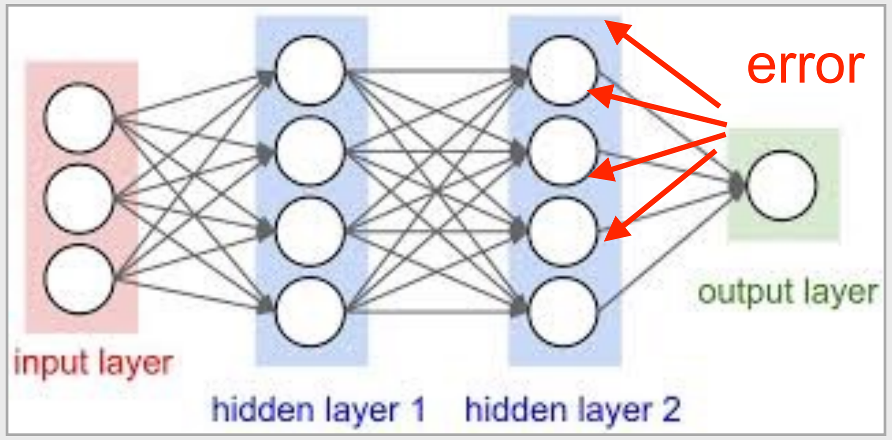

AMNs can be *trained* through a modification of the back-propagation algorithm
(i.e. weak derivatives of μ can be efficiently computed).

Sounds like a rehash of neural networks?

They are similar but…

AMNs can express discontinuous functions
whereas classical (continuous) neural networks cannot.

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a switched system	its AMN representation
$x(t+1) = \begin{cases} A^{-}x(t), & \text{if } x_1(t) \leq 0, \\ A^{+}x(t), & \text{otherwise,} \end{cases}$	$\varphi^{\text{sw}}(x) = \mu(A^{-}x, A^{+}x, e_1^T x)$

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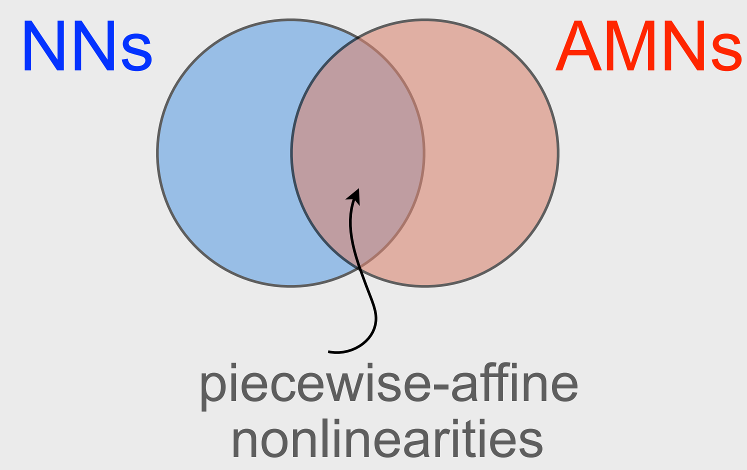
AMNs can be encoded as satisfiability modulo theory (SMT) instances.

Fine, but does the difference matter?

The answer depends on what you want to do.



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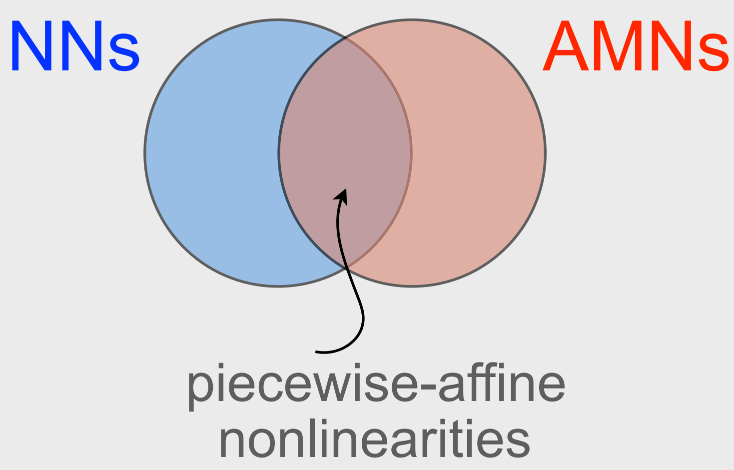
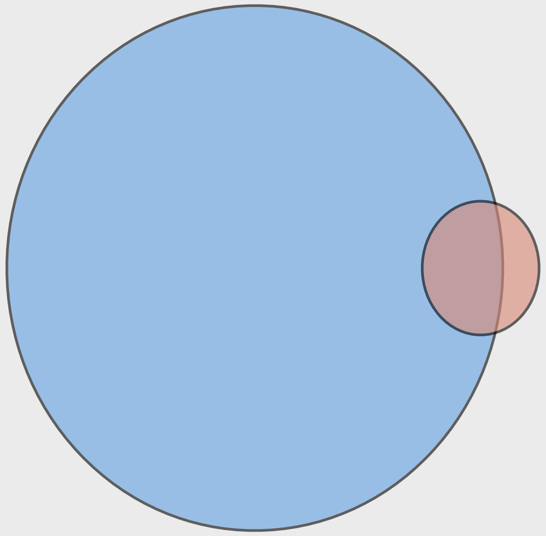


Learning view:

"Can train so much more"



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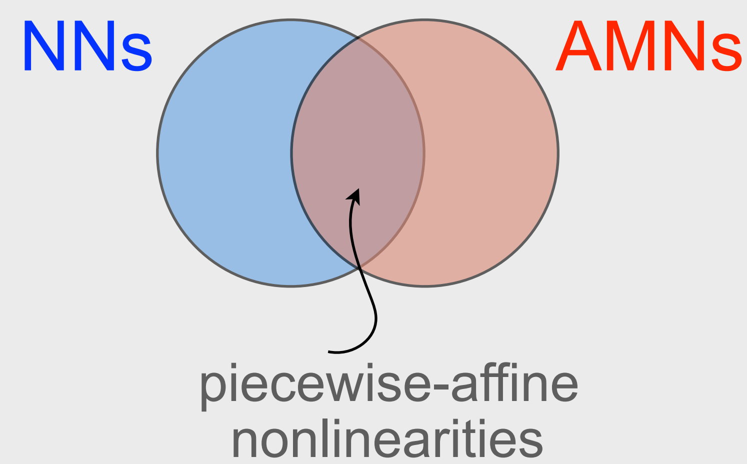

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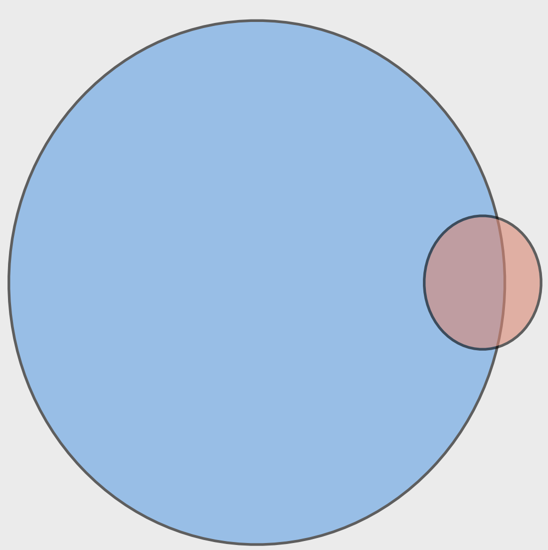
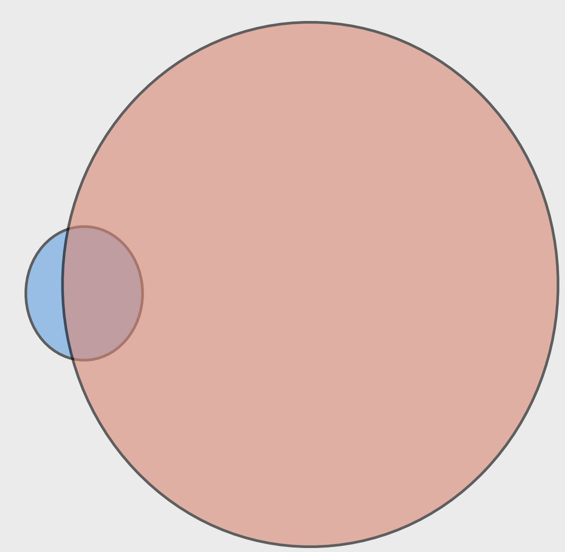


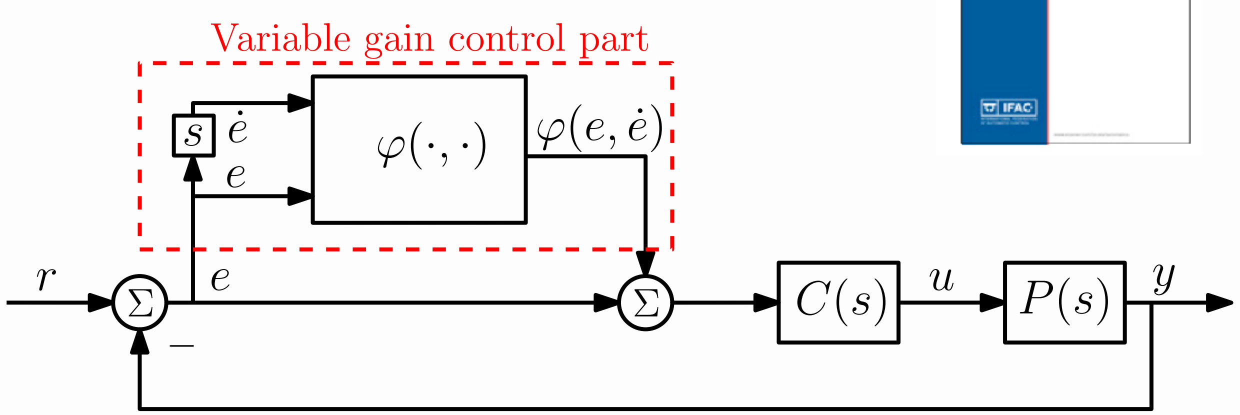
Verification view:
"Can do computational analysis"



Beating fundamental limitations of linear feedback

Overcoming a fundamental time-domain performance limitation by nonlinear control

Hunnekens, Wouw, Nesic (2016)



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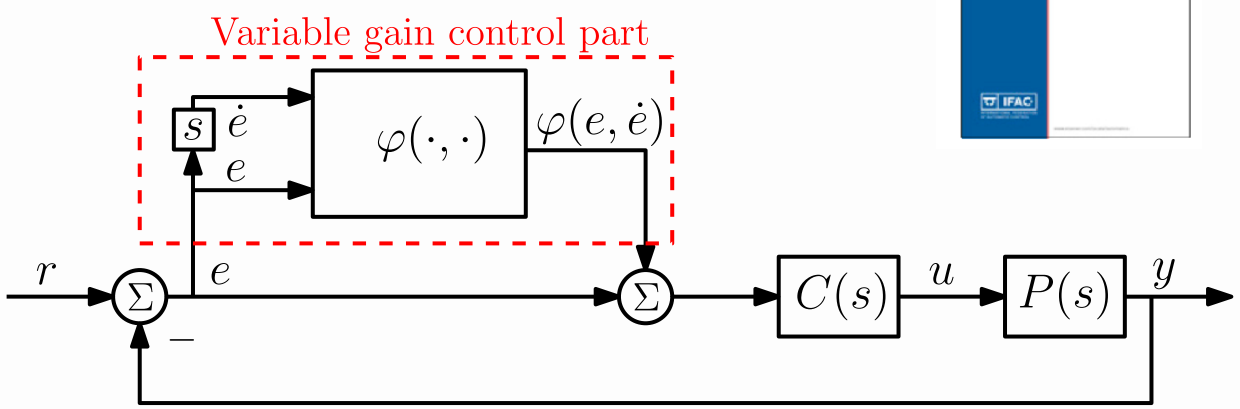

Seron, et al. (1997)

$$y_{os} \geq \frac{(pt_r - 1)e^{pt_r} + 1}{pt_r} \geq \frac{pt_r}{2}\,.$$

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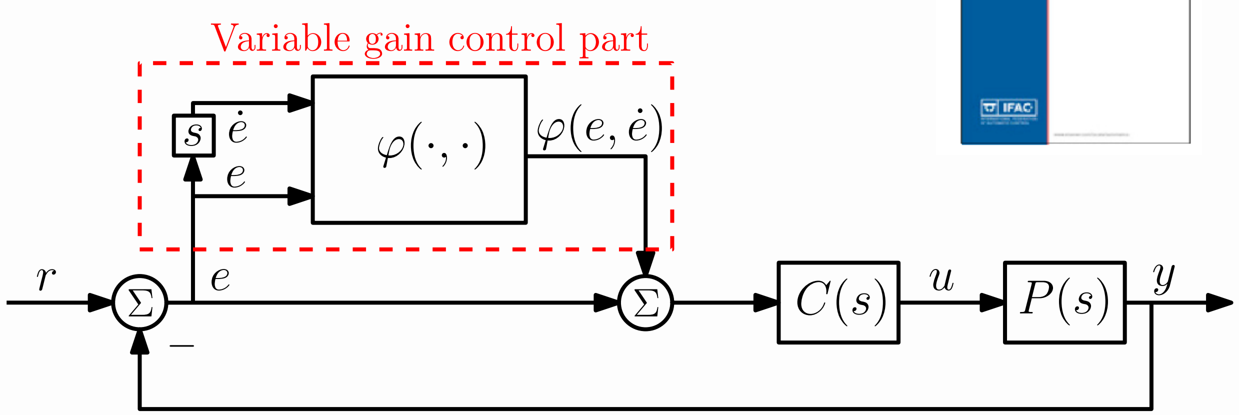
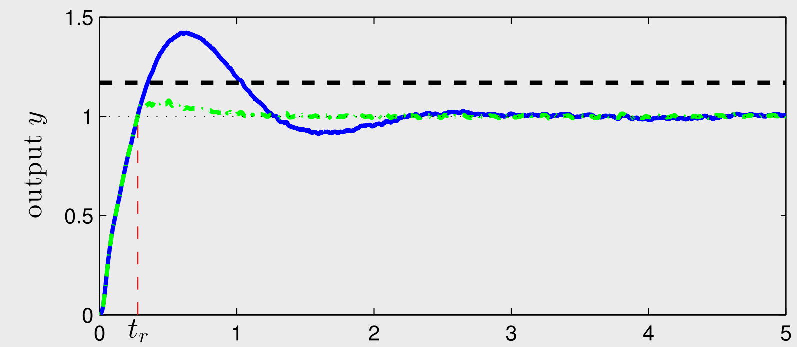


Phase-based,
variable-gain nonlinearity

$$\varphi^{\text{vgc}}(e,\dot{e}) = \begin{cases} \alpha e, & \text{if } e\dot{e} > 0 \\ 0, & \text{otherwise} \end{cases}$$



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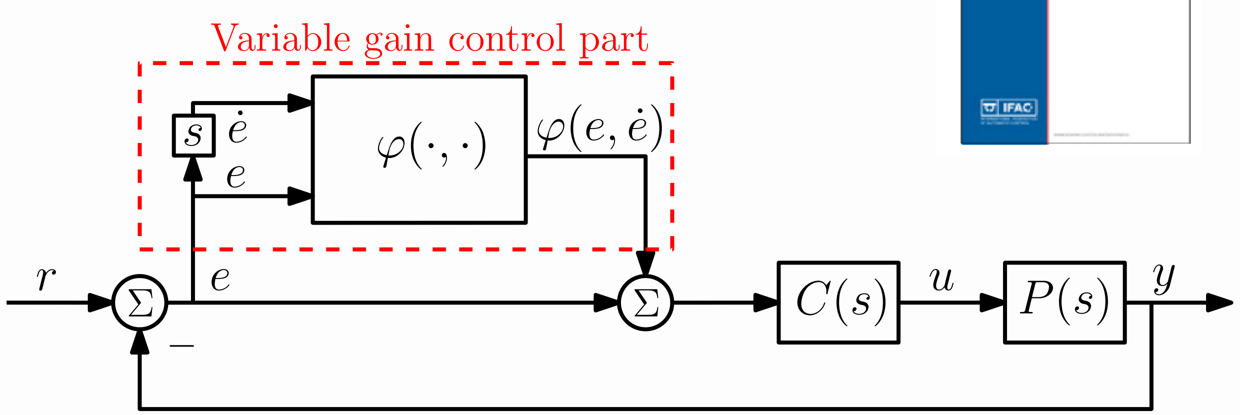

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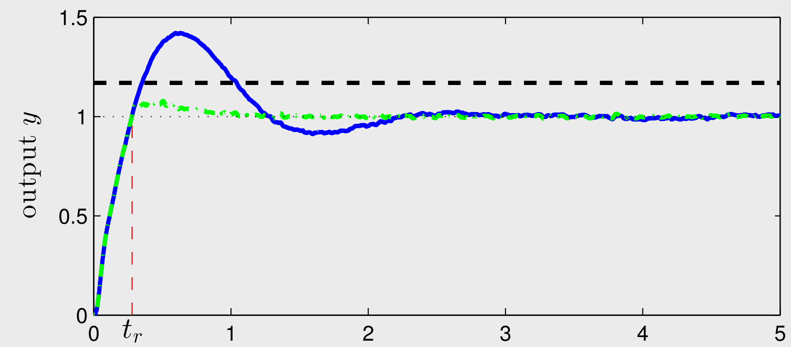
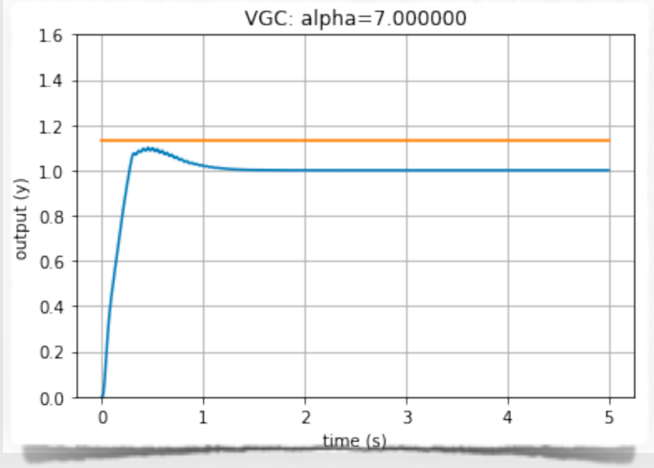


The nonlinearity as an AMN (with 4 multiplexing nonlinearities)

$$\varphi^{\text{vgc}}(e,\dot{e}) = \varphi^{\vee}\left(\varphi^{\vee}(0, \alpha e, -e, -\dot{e}), \alpha e, e, \dot{e}\right)$$

$$\varphi^{\vee}(x, y, z_1, z_2) = \mu(x, \mu(x, y, z_1), z_2)$$



What is SMT (satisfiability modulo theories)?

- Essentially, **constraint solving**
- A bad name for a useful and **extremely practical** idea.
- Burgeoning and vibrant area of CS research

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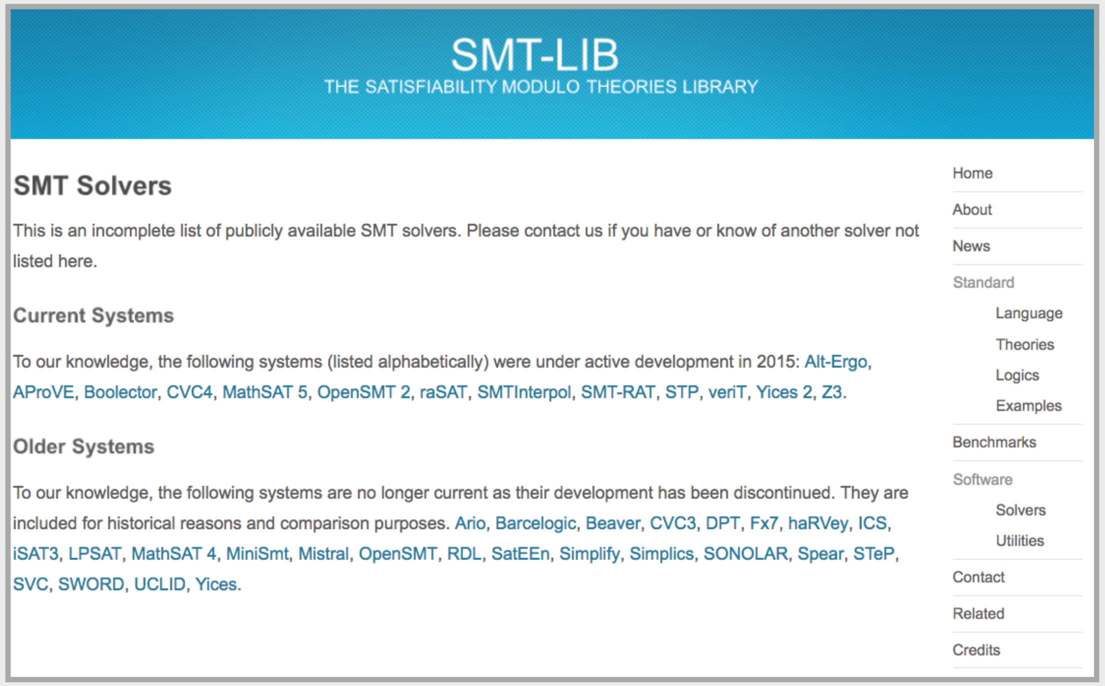
smt-lib.org



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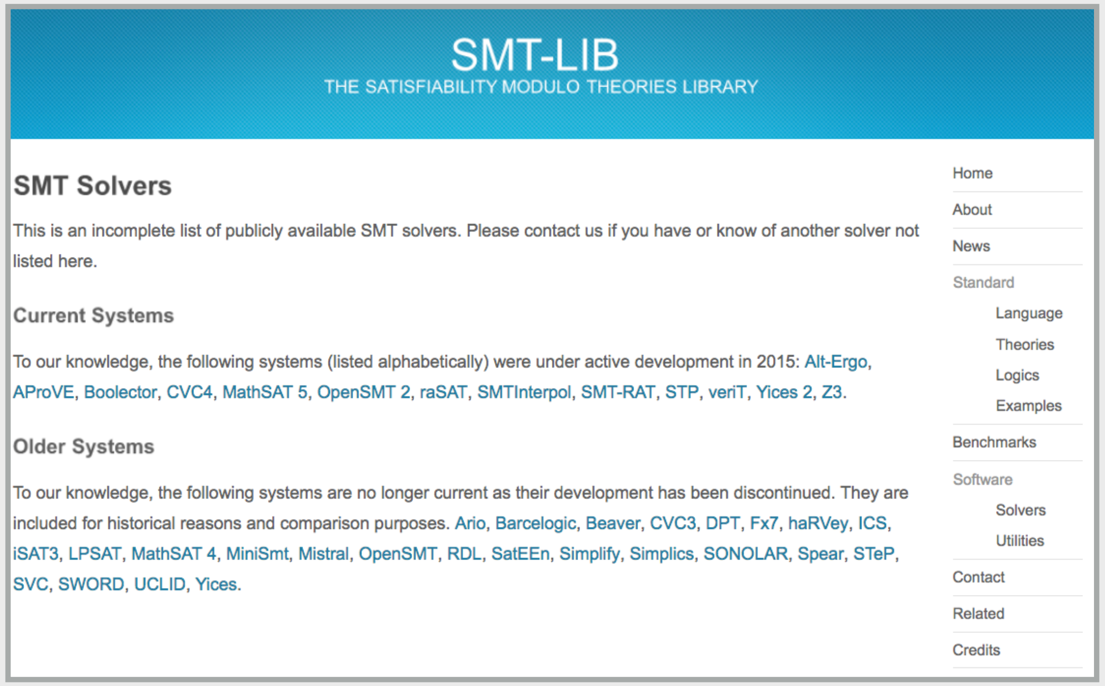
SMT-LIB
THE SATISFIABILITY MODULO THEORIES LIBRARY

SMT Solvers

This is an incomplete list of publicly available SMT solvers. Please contact us if you have or know of another solver not listed here.

Current Systems

To our knowledge, the following systems (listed alphabetically) were under active development in 2015: Alt-Ergo, AProVE, Boolector, CVC4, MathSAT 5, OpenSMT 2, raSAT, SMTInterpol, SMT-RAT, STP, veriT, Yices 2, Z3.

Older Systems

To our knowledge, the following systems are no longer current as their development has been discontinued. They are included for historical reasons and comparison purposes. Ario, Barcelogic, Beaver, CVC3, DPT, Fx7, haRVey, ICS, iSAT3, LPSAT, MathSAT 4, MiniSmt, Mistral, OpenSMT, RDL, SatEEn, Simplify, Simplics, SONOLAR, Spear, STeP, SVC, SWORD, UCLID, Yices.

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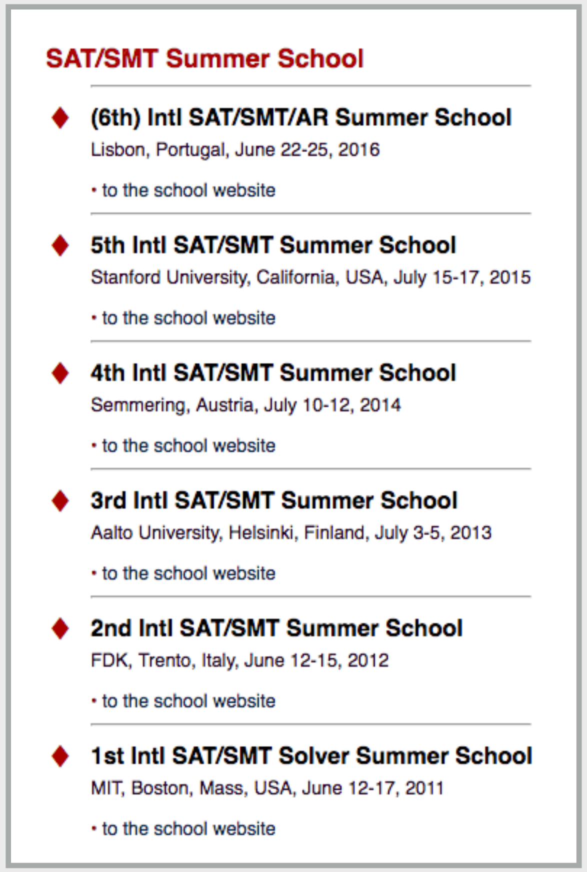
SAT/SMT Summer School

- **(6th) Intl SAT/SMT/AR Summer School**
  Lisbon, Portugal, June 22-25, 2016
  • to the school website
- **5th Intl SAT/SMT Summer School**
  Stanford University, California, USA, July 15-17, 2015
  • to the school website
- **4th Intl SAT/SMT Summer School**
  Semmering, Austria, July 10-12, 2014
  • to the school website
- **3rd Intl SAT/SMT Summer School**
  Aalto University, Helsinki, Finland, July 3-5, 2013
  • to the school website
- **2nd Intl SAT/SMT Summer School**
  FDK, Trento, Italy, June 12-15, 2012
  • to the school website
- **1st Intl SAT/SMT Solver Summer School**
  MIT, Boston, Mass, USA, June 12-17, 2011
  • to the school website

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smt-lib.org

fomlas2018.fortiss.org

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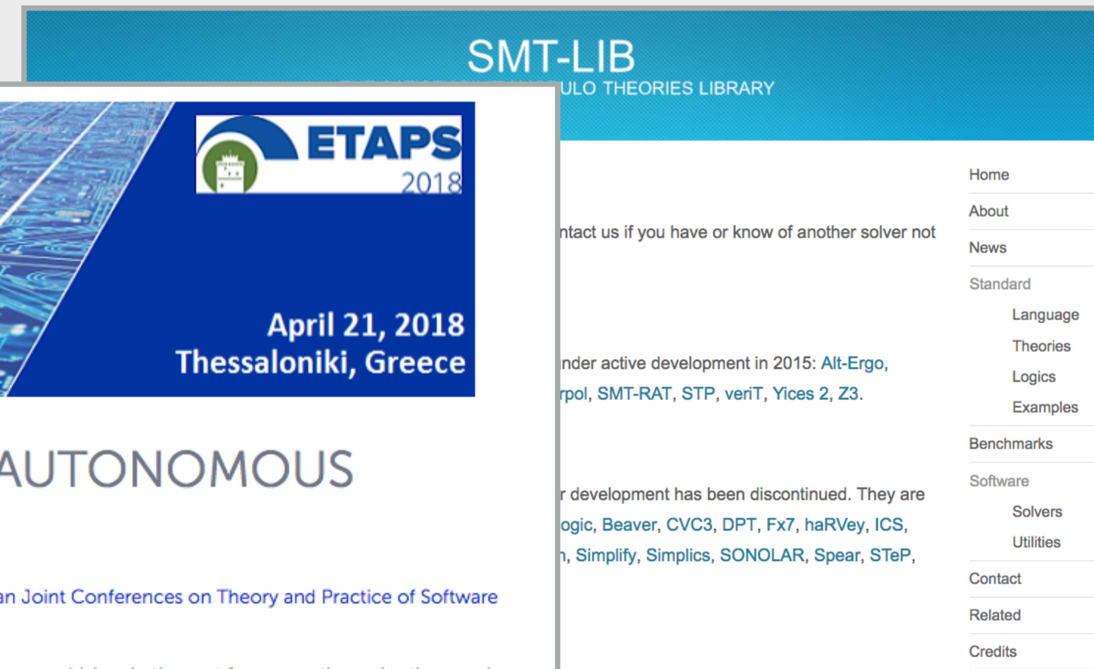
smt-lib.org



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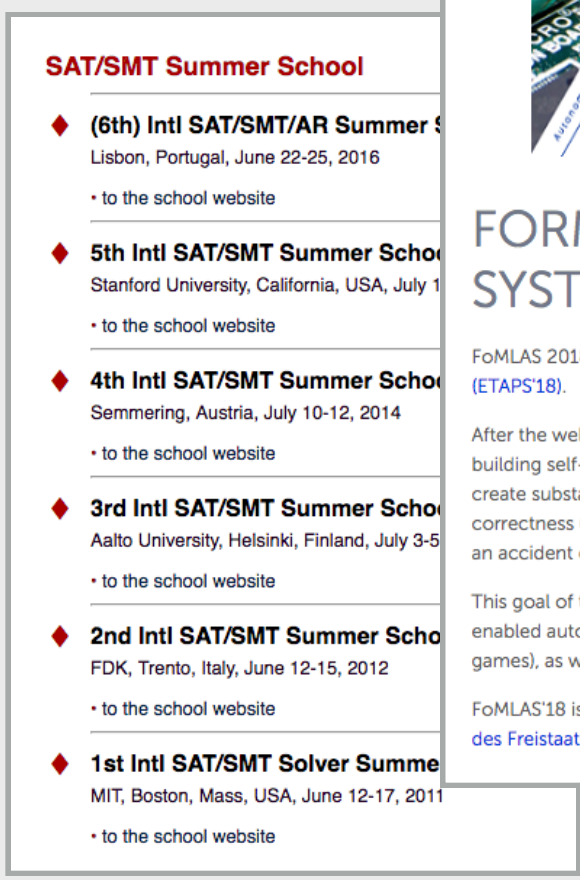

Chih-Hong Cheng
Fortiss (Germany)

Indranil Saha
IIT Kanpur (India)

TÉCNICO LISBOA
SAT Association
Microsoft Research

What is SMT (satisfiability modulo theories)?

$$((x_1 + 2x_3 < 5) \vee \neg(x_3 \leq 1) \wedge (x_1 \geq 1))$$

satisfiability: Boolean combination

of

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Linear programming solver

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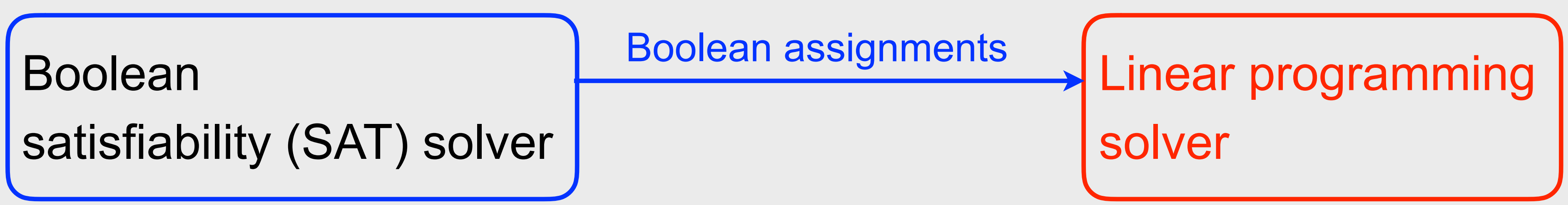

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Boolean satisfiability (SAT) solver $\xrightarrow{\text{Boolean assignments}}$ Linear programming solver

Linear programming solver $\xrightarrow{\text{Variable assignments or counterexamples}}$ Boolean satisfiability (SAT) solver

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Boolean satisfiability (SAT) solver

Boolean assignments →

← Variable assignments or counterexamples

Linear programming solver

State-of-the-art SAT solvers: Brute force yet can handle gigantic problem instances

Nature

MATHEMATICS

Maths proof smashes size record

Supercomputer produces a 200-terabyte proof — but is it really mathematics?

Wikipedia

Nevertheless, as of 2016, heuristical SAT-algorithms are able to solve problem instances involving tens of thousands of variables and formulas consisting of millions of symbols,[1]

Future and why (I think) formal methods (will) matter

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Feedback formalized
Bode, Nyquist, Black

1930s-40s

en.wikipedia.org/wiki/PID_controller

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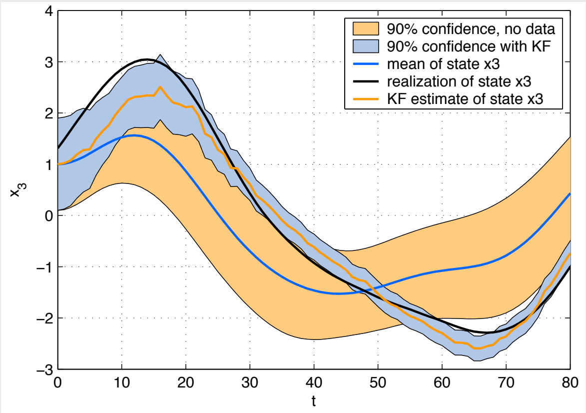

State space, filtering
(Wiener, Bellman, Kalman)

$$\hat{x}(t \mid s) = \mathbf{E}\left(x(t) \mid y(0), \ldots, y(s)\right),$$
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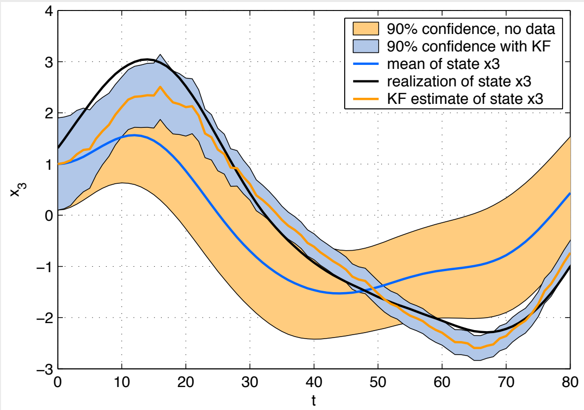

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(robust control, adaptive control
KYP, LMIs, convex optimization)

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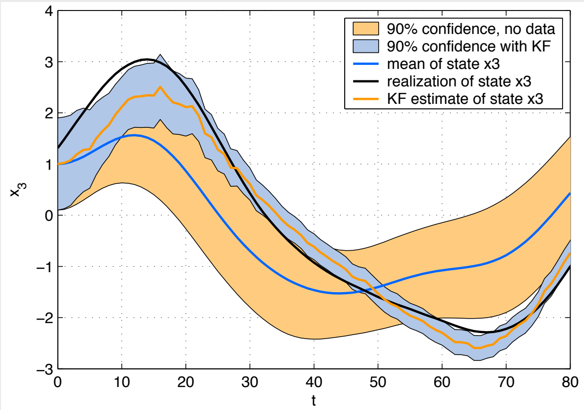

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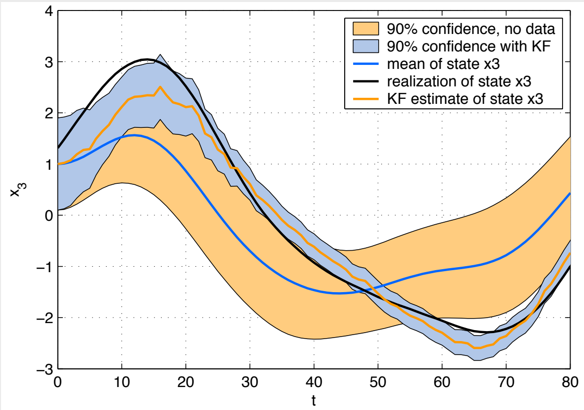

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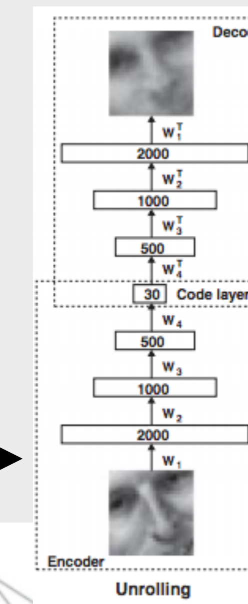

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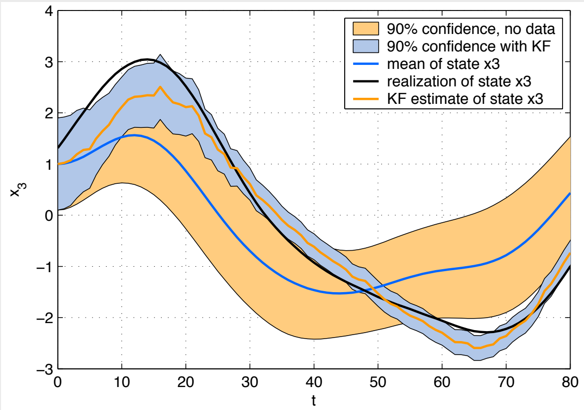

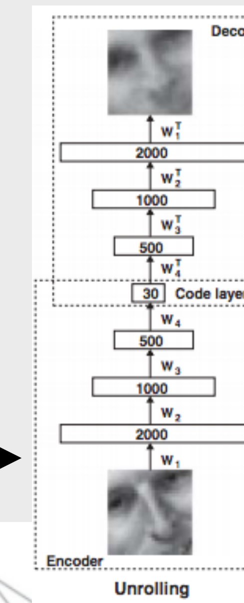

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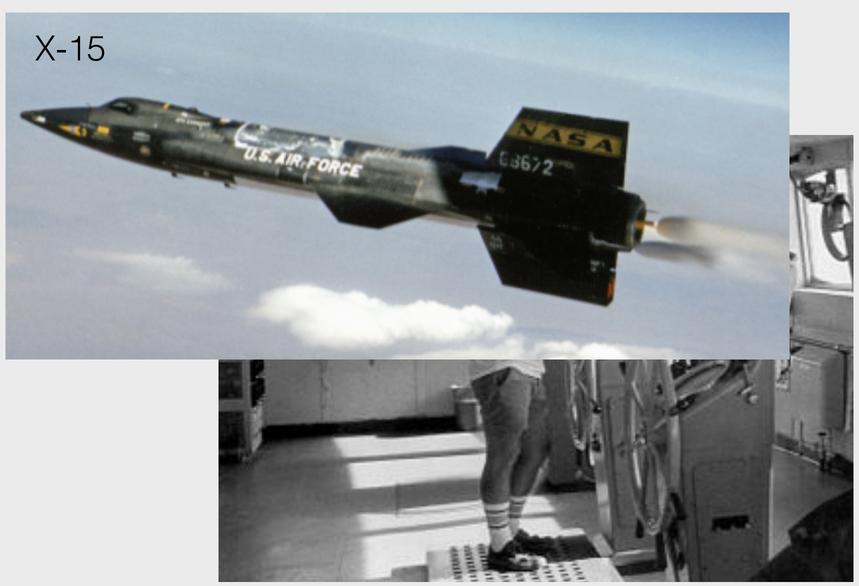

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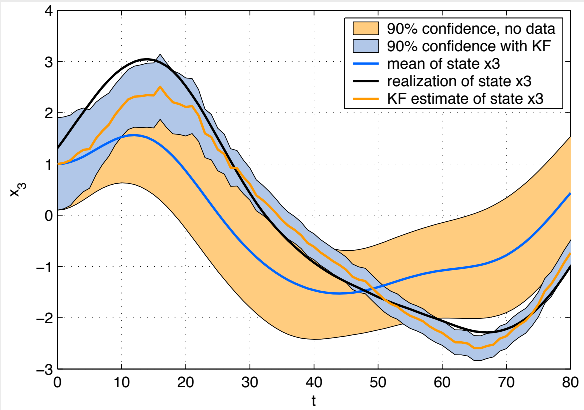

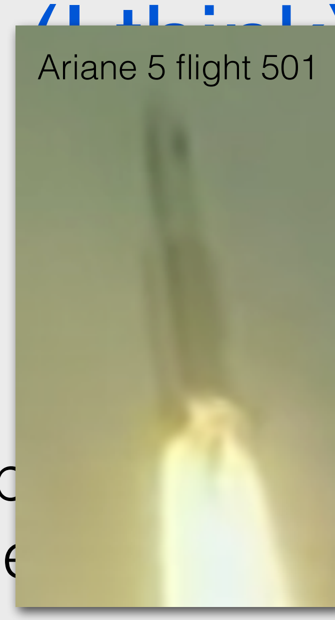

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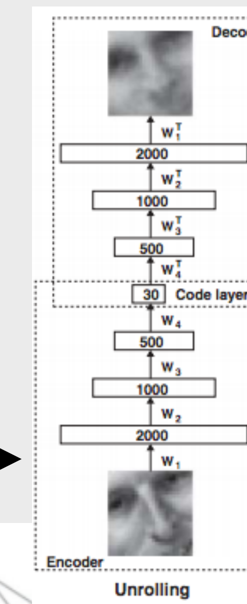

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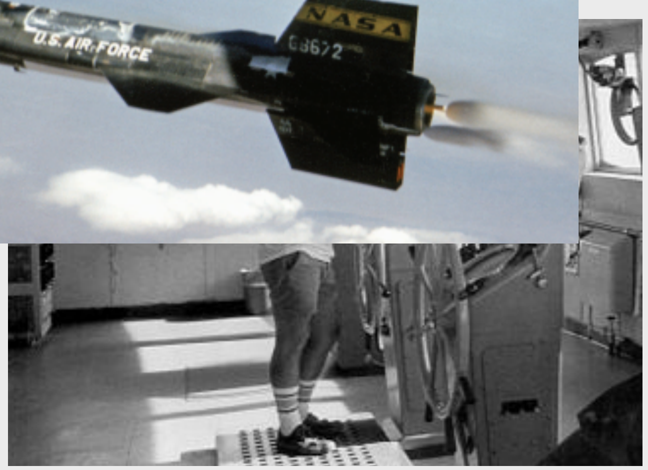

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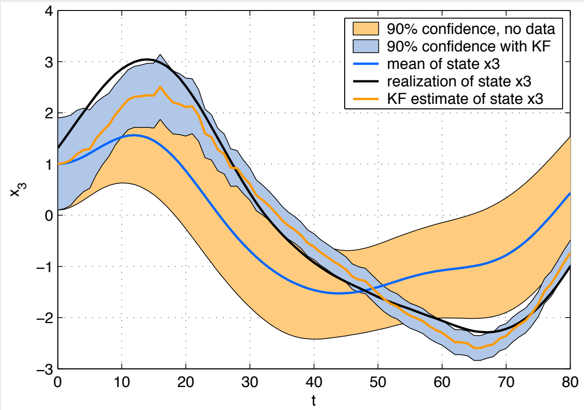
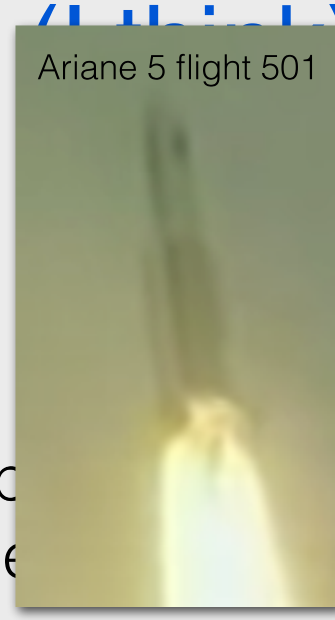
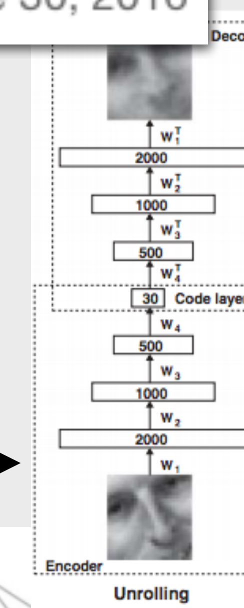




A Tragic Loss

The Tesla Team • June 30, 2016



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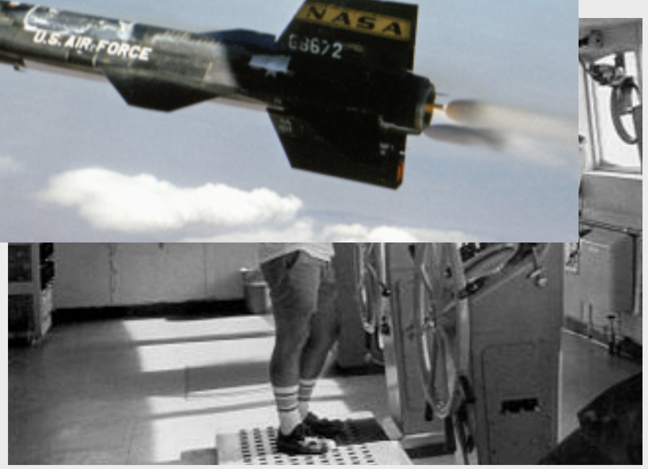

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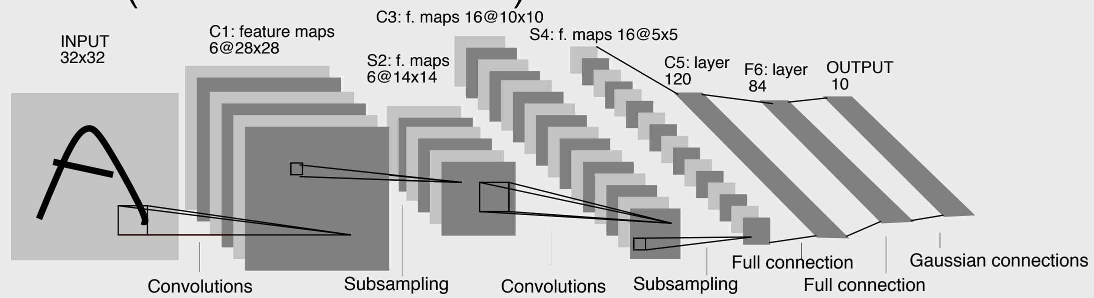
Verifying vision algorithms

LeNet-5 (Lecun et al. 1998)



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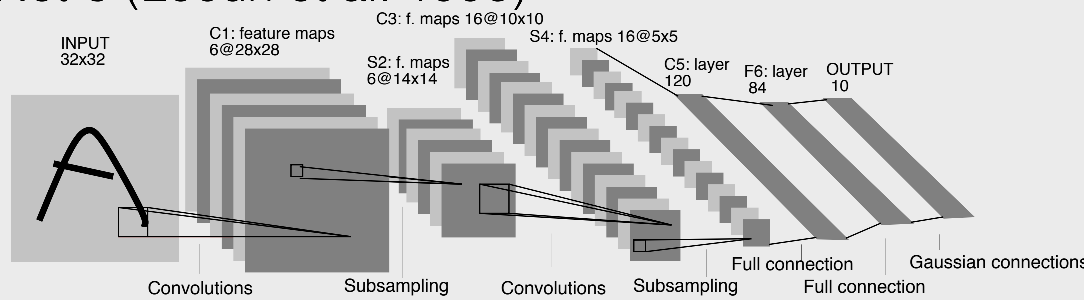
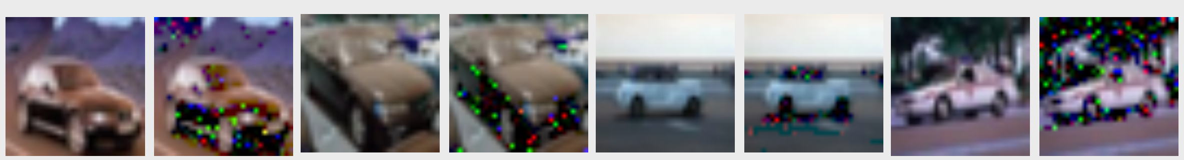


Safety Verification of Deep Neural Networks*

Xiaowei Huang, Marta Kwiatkowska, Sen Wang and Min Wu

Department of Computer Science, University of Oxford



automobile to bird   automobile to frog   automobile to airplane   automobile to horse

Fig. 1. Automobile images (classified correctly) and their perturbed images (classified wrongly)

Fig. 10. Street sign images. Found an adversarial example for the left image (class changed into bird house), but cannot find an adversarial example for the right image for 20,000 dimensions.

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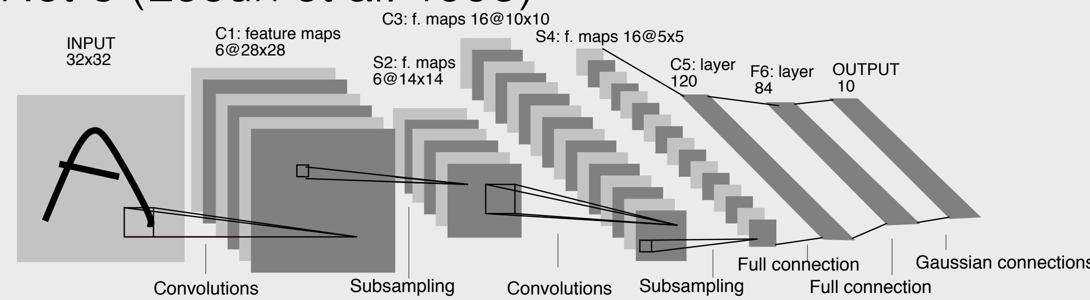

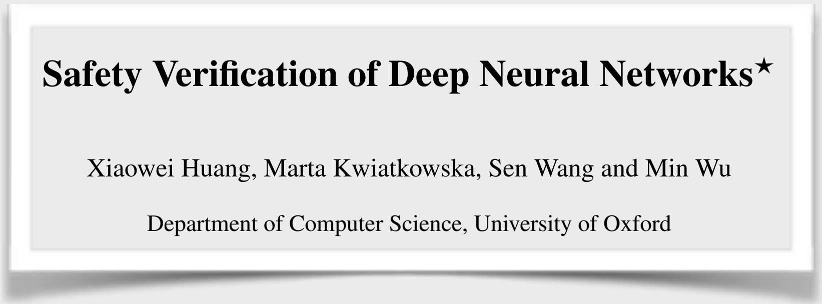

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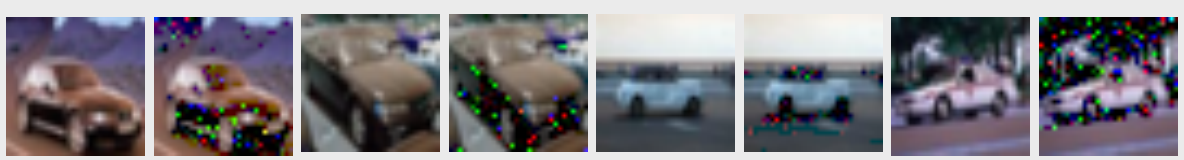

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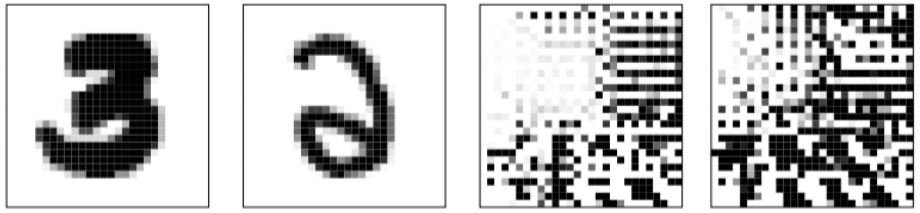
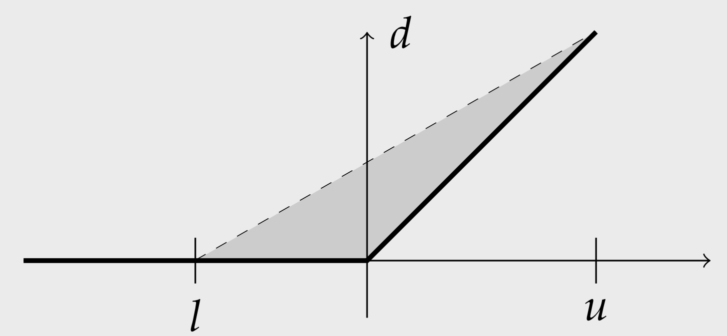
Formal Verification of Piece-Wise Linear Feed-Forward Neural Networks

Rüdiger Ehlers



(a) '3' digit from the MNIST dataset (b) '2' digit from the MNIST dataset (c) Image classified as digit 2 with $\delta = 20$ (d) Image classified as digit 2 with $\delta = 30$



linear over-approximation of ReLU activation function

SMT encoding of an AMN

Variable (identity map)

$$\boxed{\mathrm{SMT}_x[x, y] \equiv \{y = x\}}$$

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Affine
transformation

$$\alpha(\xi) = W\xi + b,$$

$$\mathrm{SMT}_{\alpha(\varphi_1)}[x, y] \equiv \left\{ \begin{array}{r} \exists v.\,(y = Wv + b) \\ \land\ \mathrm{SMT}_{\varphi_1}[x, v] \end{array} \right\}$$

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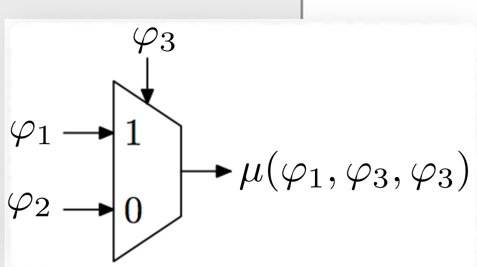
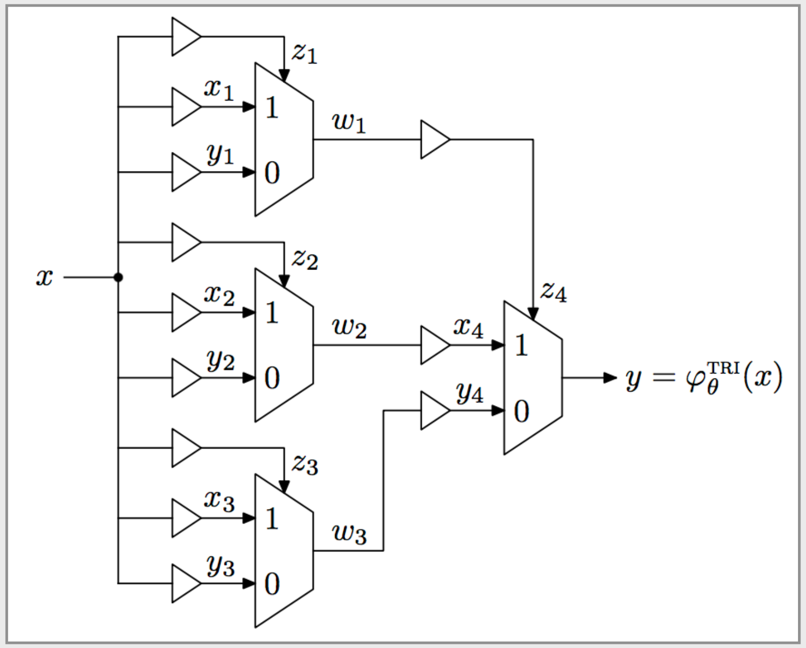
Multiplexing function

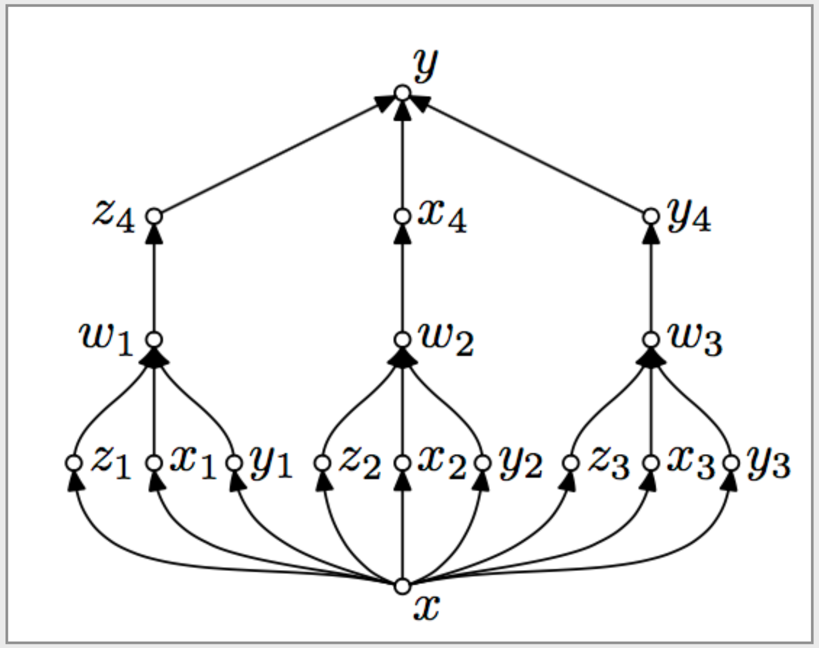
$$\mathrm{SMT}_{\mu(\varphi_1,\varphi_2,\varphi_3)}[x, y] \equiv \left\{ \begin{array}{l} \exists u, v, w.\,((w \leq 0) \to (y = u)) \\ \qquad\qquad \wedge\ (\neg(w \leq 0) \to (y = v)) \\ \qquad\qquad \wedge\ \mathrm{SMT}_{\varphi_1}[x, u] \\ \qquad\qquad \wedge\ \mathrm{SMT}_{\varphi_2}[x, v] \\ \qquad\qquad \wedge\ \mathrm{SMT}_{\varphi_3}[x, w] \end{array} \right\}$$



SMT encoding of a “triplexer”

Triplexer

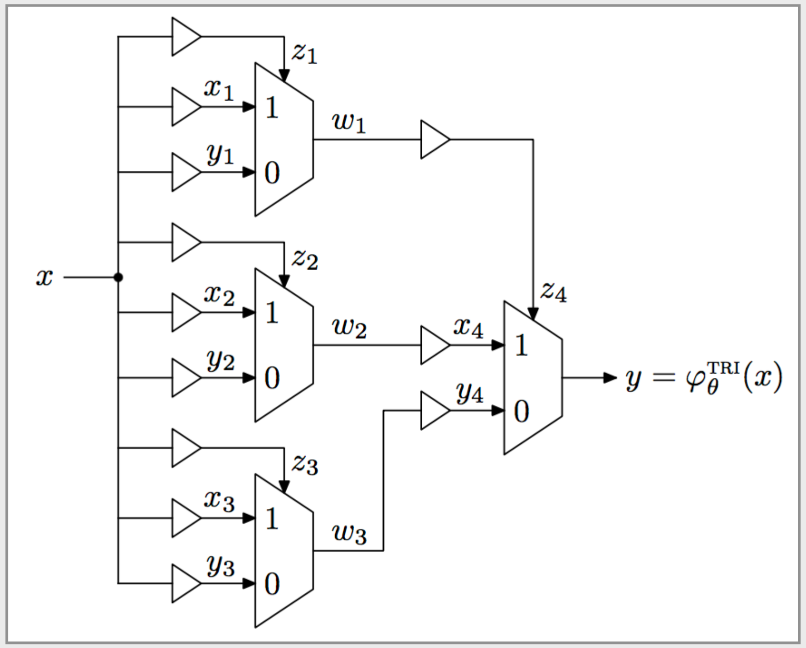


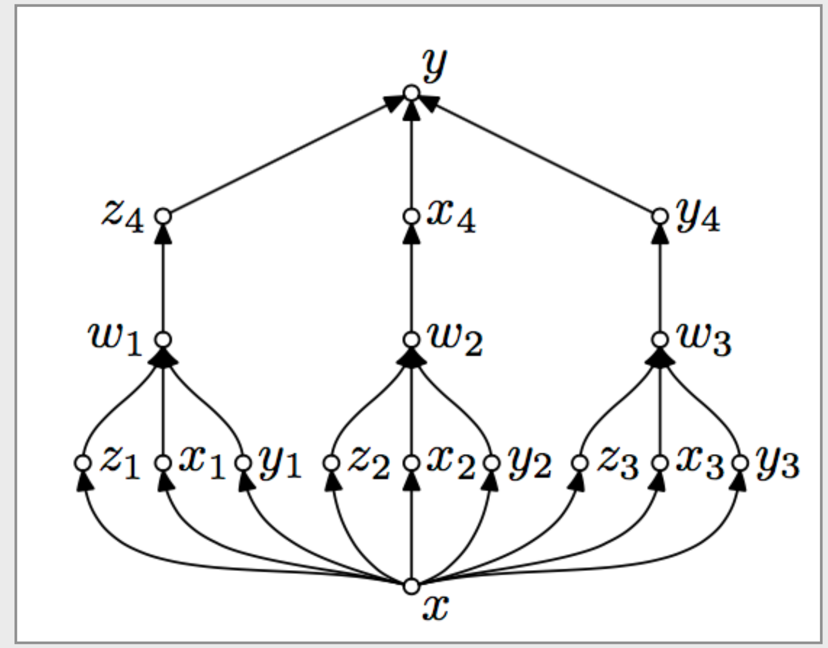


Computation graph

SMT encoding of a “triplexer”

Triplexer





Computation graph

SMT encoding

$$\mathrm{SMT}_{\varphi_\theta^{\mathrm{TRI}}}[x,y] \equiv \left\{ \begin{array}{l} \exists (x_1, y_1, z_1, \ldots, x_4, y_4, z_4, w_1, w_2, w_3) \in \mathbf{R}^{15}. \\ \quad \displaystyle\bigwedge_{i=1}^{3} (x_i = a_i x + b_i \wedge y_i = c_i x + d_i \wedge z_i = e_i x + f_i) \\ \quad \wedge \displaystyle\bigwedge_{j=1}^{3} ((z_j \leq 0) \rightarrow (w_j = x_j)) \wedge (\neg(z_j \leq 0) \rightarrow (w_j = y_j)) \\ \quad \wedge (x_4 = a_4 w_2 + b_4 \wedge y_4 = c_4 w_3 + d_4 \wedge z_4 = e_4 w_1 + f_4) \\ \quad \wedge ((z_4 \leq 0) \rightarrow (y = x_4)) \wedge (\neg(z_4 \leq 0) \rightarrow (y = y_4)) \end{array} \right\}$$

System analysis with AMNs in the loop

discrete-**time** system
with
an AMN φ

$$x^+ = x(t+1) = \varphi(x(t)), \quad x(0) = x_0, \quad t = 0, 1, 2, \ldots,$$

System analysis with AMNs in the loop

discrete-**time** system
with
an AMN φ

$$x^+ = x(t+1) = \varphi(x(t)), \quad x(0) = x_0, \quad t = 0, 1, 2, \ldots,$$

Global asymptotic stability: If there exists V: $R^n \rightarrow R$ such that

$$(V(0) = 0) \ \wedge \ (x \neq 0 \rightarrow V(x) > 0) \ \wedge \ (V(x^+) - V(x) < 0), \ldots$$

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General idea: Pick a class of V's such that the sufficient conditions can be expressed as an SMT (e.g., with linear arithmetic as the theory).

An example:

$$V(x) = \max_i \ \{a_i^T x + b_i\}$$

Yet, any AMN-representable *V* will do, too.

Search for Lyapunov certificates

Want to solve: $$\exists V \in \mathcal{V}.\, \forall x \in \mathcal{X}.\ \textsc{lyap}(V, x)$$

there exists *V*
satisfying the sufficient conditions
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Counterexample-guided search:

algorithm: Counterexample guided Lyapunov search
initialize: $k := 0$, $x_0 \in \mathcal{X}$, $\mathcal{X}_0 := \{x_0\}$
repeat:
1. *Search for a candidate Lyapunov function.*
   if $\text{E-}\textsc{solve}(\mathcal{X}_k)$ **then:**
   $V_k :=$ solution to $\text{E-}\textsc{solve}(\mathcal{X}_k)$
   else return FALSE/UNKNOWN
2. *Generate counterexample.*
   if $\text{F-}\textsc{solve}(V_k)$ **then:**
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3. *Update counterexample set.*
   $\mathcal{X}_{k+1} := \mathcal{X}_k \cup \{x_{k+1}\}$
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until: stopping criterion

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Generate candidate *V*
(LYAP imposed over a subset of the states)

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$$\text{F-SOLVE}(V) \equiv \exists x \in \mathcal{X}.\, \neg\textsc{lyap}(V, x)$$

Find states at which LYAP is violated.
Revise the conditions for step (1)

Sanity check

A randomly generated linear system
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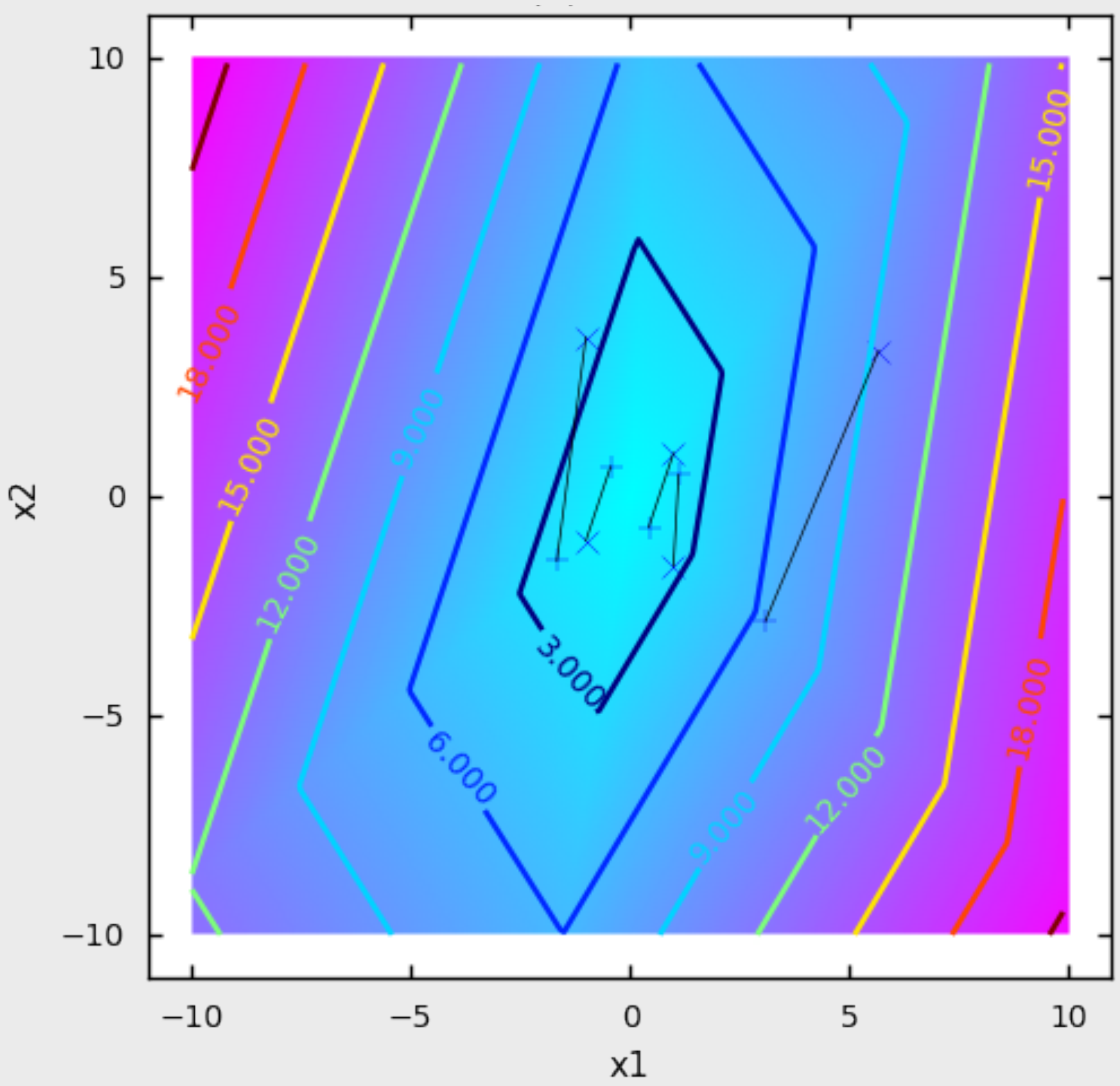
5 iterations of the algorithm constructs a Lyapunov function.

Each iteration adds a “face” to *V*.

$$V(x) = \max_i \; \{a_i^T x + b_i\}$$



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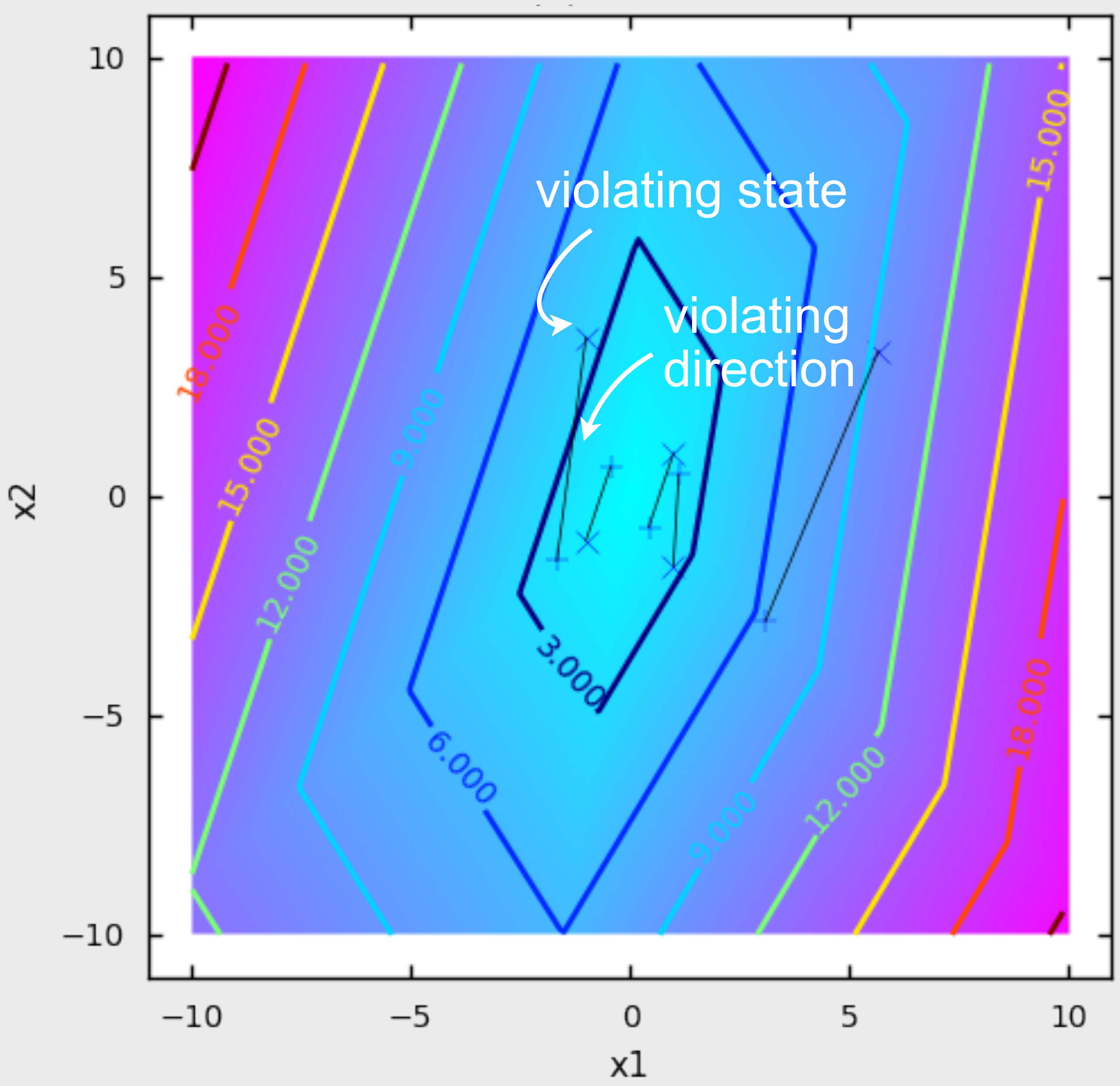
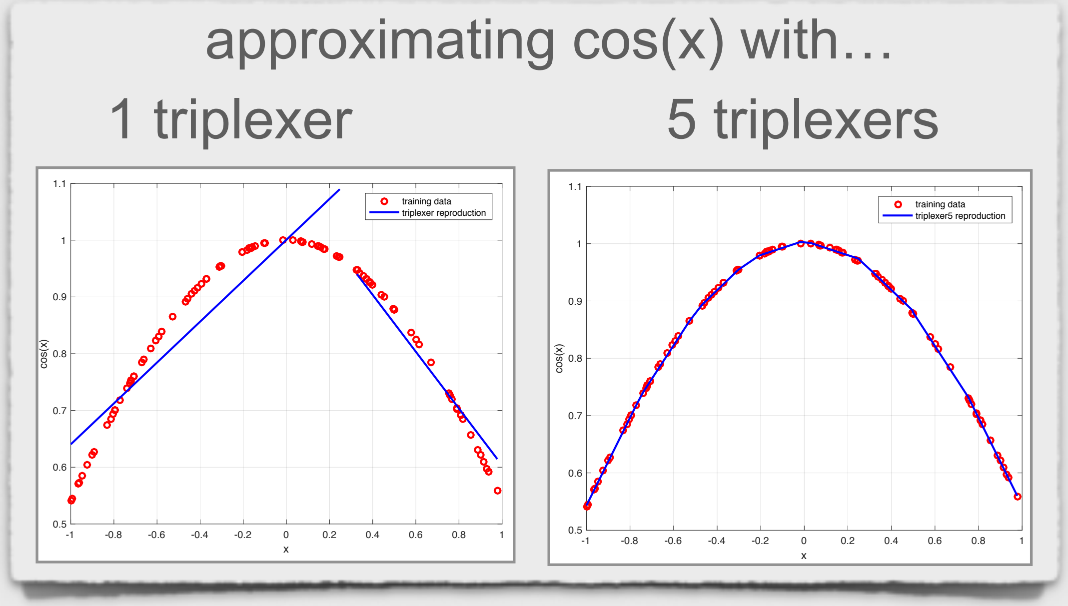


Completeness of AMNs in function approximation

Completeness over continuous functions is inherited from similar completeness of neural networks.



Given a compact subset $X \subset \mathbf{R}^n$ and a continuous function $f : X \to \mathbf{R}$, there exists an affine multiplexing network $\varphi : X \to \mathbf{R}$ that approximates f arbitrarily well on X in L_2-norm.

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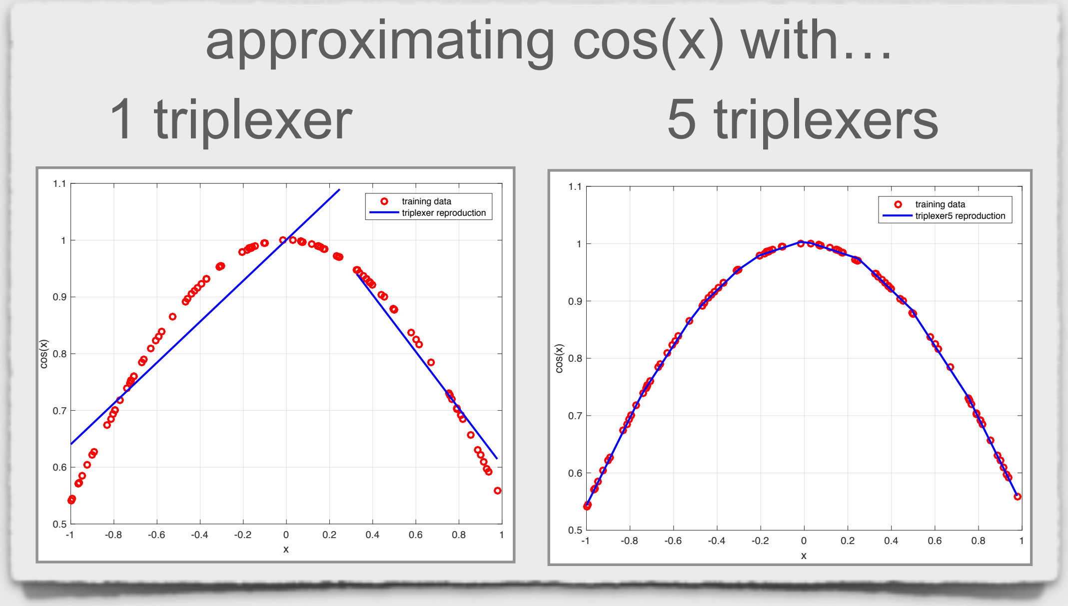

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Open question: Can the added expressivity of AMNs be utilized for completeness over a broader family of functions?

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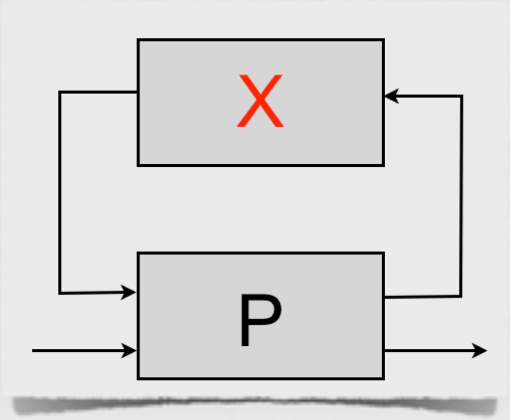
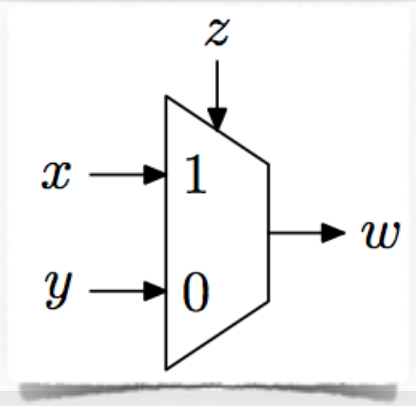
- Affine multiplexing networks
- Connections to system analysis, learning and decision procedures





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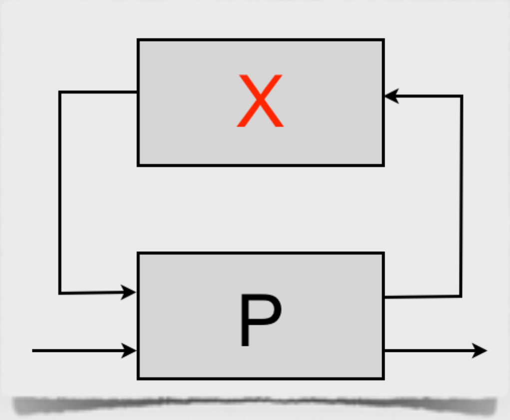

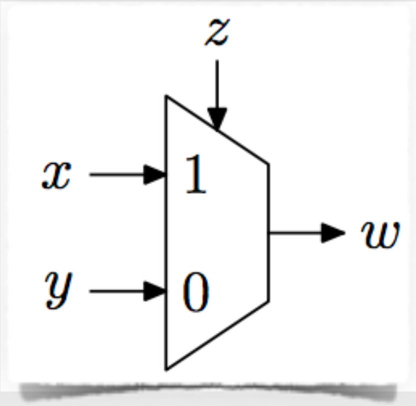

A few (potential) **future steps**

- Alternative means for nonlinear control synthesis and beating fundamental limitations
- Training Lyapunov-type certificates in AMN form directly from data (i.e., simulations)
- Continuous-time dynamics (modulo technicalities)
- Control-oriented theories in SMT
- Domain-specific languages

$$\mu_{\mathcal{K}}(x,y,z) = \begin{cases} x, & \text{if } z \in \mathcal{K}, \\ y, & \text{otherwise.} \end{cases}$$

arbitrary cone
(AMN: $z \leq 0$ for $z \in \mathbb{R}$)

Difficulties

- At its heart, involves at least (worst-case) an **NP-complete** computation
- Model-based (IQCs, robust control)
- Some good tools exist (Z3, yices), but these require expert knowledge to use
- Writing a new Python toolbox (AMNET) as a modeling and query layer

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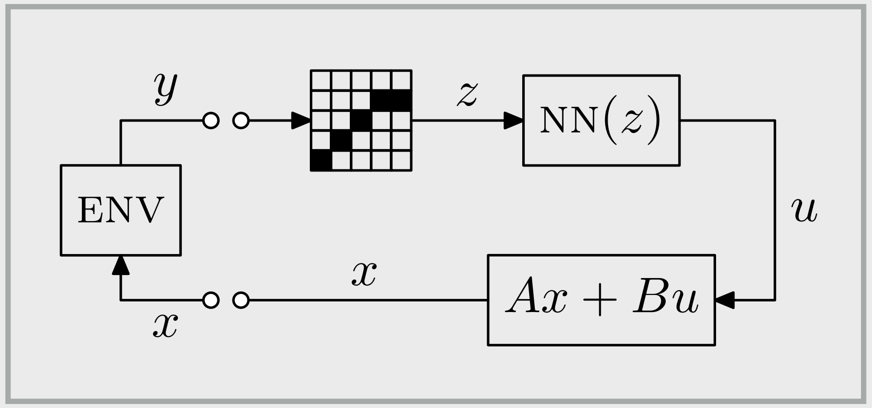

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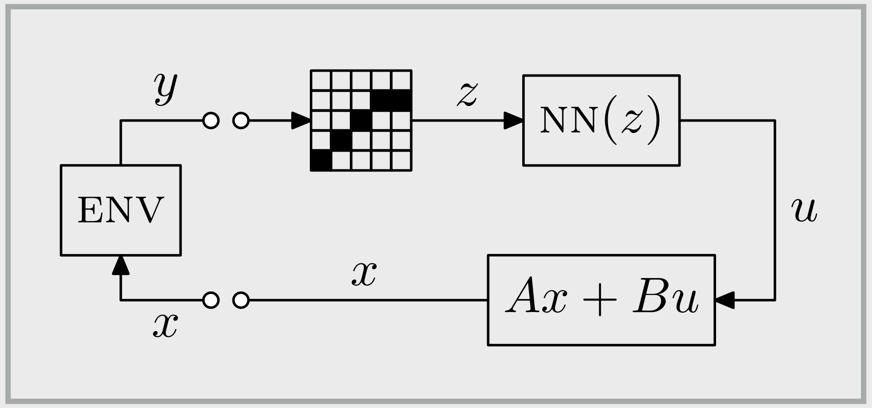

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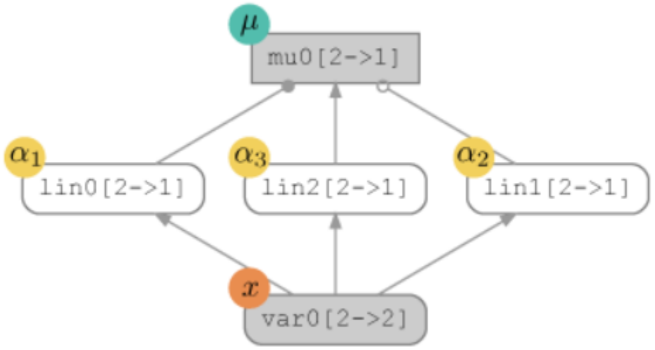
https://github.com/ipapusha/amnet

AMNET: Affine Multiplexing Network Toolbox

AMNET is a Python toolbox that assists in building certain kinds of neural networks, and formally verifying their behavior in-the-loop (under development).



1. Multiplexing nonlinearities

$$\mathtt{max2}(x) = \mu(\alpha_1(x), \alpha_2(x), \alpha_3(x))$$
$$= \mathtt{if}\ (-x_1 + x_2 \le 0)\ \mathtt{then}\ x_1\ \mathtt{else}\ x_2$$

2. Affine transformations

$$\alpha_1(x) := e_1^T x \quad \alpha_2(x) := e_2^T x \quad \alpha_3(x) := -x_1 + x_2$$

3. Input variables

$$x \in \mathbf{R}^2$$

```
import numpy as np
from amnet import Variable, Linear, Mu

# a two-dimensional input variable
x = Variable(2, name='x')

# choose components
a1 = Linear(np.array([[1, 0]]), x)
a2 = Linear(np.array([[0, 1]]), x)

# find difference
a3 = Linear(np.array([[-1, 1]]), x)

# if a3 <= 0, returns a1; otherwise a2
phimax = Mu(a1, a2, a3)

# equivalently, we can also write
# phimax = amnet.atoms.max_all(x)

print phimax
print phimax.eval([1, -2]) # returns: 1
```