Ivan Papusha 05/27/2015 15:42

CDS270-2 Exercises

1. *Block matrix gymnastics.* In this problem we will derive classical block matrix "tricks" using just linear algebra. (If you need to take a derivative, you are doing it wrong!) Let $A = A^T \in \mathbf{R}^{n\times n}$ and $D = D^T \in \mathbf{R}^{m\times m}$ be symmetric matrices, and $B \in \mathbf{R}^{n\times m}$.

   (a) *Matrix completion of squares.* Show that if A is invertible, then

   $$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (x + A^{-1}By)^T A (x + A^{-1}By) + y^T(D - B^T A^{-1} B)y$$

   for all $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$.

   (b) Further assuming that $A \succ 0$, partially minimize over x to conclude that

   $$\inf_{x\in\mathbf{R}^n} \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = y^T(D - B^T A^{-1} B)y.$$

   The quantity $S = D - B^T A^{-1} B$ is called the *Schur complement* of A.

   (c) Use the previous results to prove that $A \succ 0$ and $S \succ 0$ if and only if

   $$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \succ 0.$$

   (d) *Block LDU decomposition.* Show that if A is invertible, then

   $$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ B^T A^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix},$$

   where S is the Schur complement of A. This decomposition is also known as the Aitken block diagonalization formula. *Hint.* start with completion of squares.

   (e) Use the block LDU decomposition to show that

   $$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}B^T A^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}B^T A^{-1} & S^{-1} \end{bmatrix},$$

   provided the inverse exists. *Hint.* what is the inverse of a block triangular matrix? block diagonal matrix? Consider the 2×2 case and generalize.

   (f) Permute the completion of squares formula and/or partially minimize over y to show the analogous formula

   $$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}^{-1} = \begin{bmatrix} T^{-1} & -T^{-1}BD^{-1} \\ -D^{-1}B^T T^{-1} & D^{-1} + D^{-1}B^T T^{-1} B D^{-1} \end{bmatrix},$$

   where $T = A - BD^{-1}B^T$ is the Schur complement of D. Clearly state any assumptions.

(g) *Matrix inversion lemma.* Provided the inverses exist, prove that

$$(A - BD^{-1}B^T)^{-1} = A^{-1} + A^{-1}B(D - B^T A^{-1} B)^{-1} B^T A^{-1}.$$

Conclude the *rank-one update* formula, which states

$$(A + bb^T)^{-1} = A^{-1} - \frac{(A^{-1}b)(A^{-1}b)^T}{1 + b^T A^{-1} b}$$

for all $b \in \mathbf{R}^n$.

(h) *Matrix determinant lemma.* Provided the inverses exist, prove that

$$\det \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} = (\det A)(\det S) = (\det D)(\det T).$$

Conclude that $\det(A + bb^T) = (1 + b^T A^{-1} b) \det A$ for all $b \in \mathbf{R}^n$.

2. *LMI characterizations of matrix eigenvalues.* Let $A \in \mathbf{R}^{m\times n}$ be a matrix with $m \geq n$, and $X = X^T \in \mathbf{R}^{n\times n}$ be a symmetric matrix (no requirement on definiteness). Prove the following:

(a) $\lambda_{\max}(X) \leq t$ if and only if $X \preceq tI$.

(b) $\lambda_{\min}(X) \geq s$ if and only if $sI \preceq X$.

(c) Assuming $t \geq 0$, prove that $\sigma_{\max}(A) \leq t$ if and only if $A^T A \preceq t^2 I$. Conclude that $\sigma_{\max}(A) \leq t$ if and only if

$$\begin{bmatrix} tI & A^T \\ A & tI \end{bmatrix} \succeq 0.$$

3. *Lyapunov exponent.* The decay rate of the system $\dot{x}(t) = f(x(t))$ is the largest (supremum) real number α such that $\lim_{t\to\infty} e^{\alpha t}\|x(t)\|_2 = 0$ for all trajectories $x(t)$ satisfying the ODE.

(a) Suppose there exists a quadratic Lyapunov function $V(z) = z^T P z$, such that

$$\frac{dV(x(t))}{dt} \leq -2\alpha V(x(t))$$

for all trajectories, where $P \succ 0$ is given. Show that the decay rate is at least α. (*Hint.* apply Grönwall's inequality.)

(b) For the LTI system $\dot{x}(t) = Ax(t)$, prove that the decay rate is equal to the *stability degree* of the matrix A, which is defined as

$$\mathrm{SD}(A) = -\max_{1\leq i\leq n} \mathrm{Re}(\lambda_i(A)).$$

In other words, the decay rate of an LTI system is the signed distance from the imaginary axis of the rightmost eigenvalue of A (positive if A is stable.)

(c) Show that the stability degree of a matrix A is at least α if there exists $P \succ 0$ such that

$$A^T P + PA + 2\alpha P \preceq 0,$$

or equivalently if there exists $Q \succ 0$ such that

$$QA^T + AQ + 2\alpha Q \preceq 0.$$

(d) Describe a method that determines the stability degree of a given matrix $A \in \mathbf{R}^{n \times n}$ by solving a sequence of LMIs. (*Hint.* Use bisection on α.) Implement your method on the specific matrix

$$A = \begin{bmatrix} -7 & 3 & -10 \\ 17 & 11 & -34 \\ 9 & 7 & -18 \end{bmatrix},$$

and check your answer by finding the eigenvalues of A.

4. *Inner restriction of LMIs.* [AM14] A square matrix $A \in \mathbf{R}^{n \times n}$ is weakly diagonally dominant if its entries satisfy

$$|A_{ii}| \geq \sum_{j \neq i} |A_{ij}|, \qquad \text{for all } i = 1, \ldots, n.$$

Define K_{dd} as the set of symmetric, weakly diagonally dominant matrices with non-negative diagonal entries,

$$K_{\mathrm{dd}} = \{A \in \mathbf{S}^n \mid A \text{ is weakly diagonally dominant}, A_{ii} \geq 0, i = 1, \ldots, n.\}$$

(a) Show that K_{dd} is a convex cone.

(b) Prove the inclusion $K_{\mathrm{dd}} \subseteq \mathbf{S}^n_+$. Is the inclusion strict?

(c) Consider the standard form semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(CX) \\ \text{subject to} & \mathbf{Tr}(A_i X) = b_i, \quad i = 1, \ldots, p \\ & X \succeq 0 \end{array}$$

with matrix variable $X \in \mathbf{S}^n$ and parameters $C, A_1, \ldots, A_p \in \mathbf{S}^n$. Suggest a linear program (LP) that gives an upper bound on the optimal value of the SDP above.

5. *SOCP restriction of LMI.* [AM14] This problem is an extension of the previous one. A matrix $A \in \mathbf{R}^{n \times n}$ is scaled diagonally dominant if there exists a diagonal matrix $D \succ 0$ such that DAD is diagonally dominant. Define the cone

$$K_{\mathrm{sdd}} = \{A \in \mathbf{S}^n \mid \exists D \succ 0, DAD \text{ is weakly diagonally dominant}, A_{ii} \geq 0, i = 1, \ldots, n.\}$$

(a) Prove the (strict) inclusions $K_{\text{dd}} \subset K_{\text{sdd}} \subset \mathbf{S}^n_+$.

(b) The *second order cone* (sometimes called the ice-cream cone) is defined as

$$\mathcal{Q}^n = \{(t, x) \in \mathbf{R} \times \mathbf{R}^{n-1} \mid \|x\|_2 \le t\}$$

According to [BCPT05, Theorem 9], scaled diagonally dominant matrices can be written as sums of positive semidefinite matrices that are nonzero on a 2×2 principal sub-matrix. For example, for $n = 3$, we have $A \in K_{\text{sdd}}$ if and only if A can be written as the sum

$$A = \begin{bmatrix} x_1 & x_2 & 0 \\ x_2 & x_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} y_1 & 0 & y_2 \\ 0 & 0 & 0 \\ y_2 & 0 & y_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & z_1 & z_2 \\ 0 & z_2 & z_3 \end{bmatrix},$$

where the sub-matrices are all positive semidefinite,

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} \succeq 0.$$

Using this fact, show how to write the constraint $A \in K_{\text{sdd}}$ as an SOCP constraint (*Hint.* rewrite the 2×2 SDP constraints as hyperbolic constraints).

(c) Suggest a tighter SOCP-based restriction of the standard form SDP

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(CX) \\ \text{subject to} & \mathbf{Tr}(A_i X) = b_i, \quad i = 1, \ldots, p \\ & X \succeq 0 \end{array}$$

than the LP restriction from the previous exercise.

6. *Hyperbolic constraints.* [LVBL98] Let x and y be real scalars and $w \in \mathbf{R}^n$ a vector.

(a) Show that

$$w^T w \le xy, \quad x \ge 0, \quad y \ge 0 \quad \Longleftrightarrow \quad \left\| \begin{bmatrix} 2w \\ x - y \end{bmatrix} \right\|_2 \le x + y.$$

(b) Use the result from part (a) to rewrite the SDP constraint

$$\begin{bmatrix} x & w^T \\ w & yI_{n \times n} \end{bmatrix} \succeq 0$$

as an SOCP constraint in the variables x, $y \in \mathbf{R}$ and $w \in \mathbf{R}^n$.

7. *Euclidean Jordan algebra.* A product $\circ$ on $\mathbf{R}^n$ ($n \ge 2$) is a function $\circ : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ that takes two real vectors and returns another real vector of the same dimension. The notation x^k means $x \circ \cdots \circ x$ (k times). We do not assume $\circ$ is associative,

$$x \circ (y \circ z) \ne (x \circ y) \circ z \quad \text{in general.}$$

Define the cone of squares K for the product $\circ$ as $K = \{x \circ x \mid x \in \mathbf{R}^n\}$. The cone of squares K is symmetric if the product $\circ$ obeys the following properties:

- *Bilinearity*: $x \circ y$ is linear in x for fixed y and vice-versa
- *Commutativity*: $x \circ y = y \circ x$
- *Jordan identity*: $x^2 \circ (y \circ x) = (x^2 \circ y) \circ x$
- *Adjoint identity*: $x^T(y \circ z) = (x \circ y)^T z$

(a) *Nonnegative orthant.* Recall that the set of real vectors with nonnegative entries,

$$\mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0, \text{for all } i = 1, \ldots, n\},$$

is called the nonnegative orthant. Show that $\mathbf{R}^n_+$ is the cone of squares with $\circ$ defined by the entrywise (Hadamard) product:

$$(x \circ y)_i = x_i y_i, \quad i = 1, \ldots, n.$$

(b) *Second order cone.* Show that the second order cone,

$$\mathcal{Q}^n = \{(x_0, x_1) \in \mathbf{R} \times \mathbf{R}^{n-1} \mid \|x_1\|_2 \leq x_0\},$$

is the cone of squares corresponding to the product

$$x \circ y = \begin{bmatrix} x^T y \\ x_0 y_1 + y_0 x_1 \end{bmatrix}.$$

(c) *Positive semidefinite cone.* Show that the positive semidefinite cone,

$$\mathbf{S}^n_+ = \{X \in \mathbf{R}^{n \times n} \mid X = X^T \text{ has nonnegative eigenvalues}\} \cong \mathbf{R}^{n(n+1)/2},$$

is the cone of squares corresponding to the symmetrized matrix product

$$X \circ Y = \frac{1}{2}(XY + YX).$$

(In this matrix case, we define the cone of squares as $\{X \circ X \mid X \in \mathbf{S}^n\}$ and use the trace inner product.)

8. *Subdifferential identity.* Let $f : \mathbf{R}^n \to \mathbf{R}$ be a (closed, proper) convex real-valued function. A vector $g \in \mathbf{R}^n$ is a subgradient of f at x if

$$f(y) \geq f(x) + g^T(y - x), \quad \text{for all } y \in \mathbf{R}^n.$$

The subdifferential $\partial f(x) \subseteq \mathbf{R}^n$ is the (possibly empty) set of all subgradients of f at x. For example, if f is also differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$ is a singleton set. The convex conjugate f^* of f is a function defined as

$$f^*(y) = \sup_{x \in \operatorname{dom} f} \left(y^T x - f(x)\right).$$

Show that these two seemingly unrelated concepts are, in fact, related by

$$y \in \partial f(x) \quad \Longleftrightarrow \quad x \in \partial f^*(y).$$

9. *Conjugate of a Lyapunov function.* If $f : \mathbf{R}^n \to \mathbf{R}$ is a function, then its convex conjugate f^* is defined as

$$f^*(y) = \sup_{x \in \operatorname{dom} f} \left(y^T x - f(x) \right).$$

   (a) Let $P \succ 0$ and define $V(x) = \frac{1}{2} x^T P x$. Show that $V^*(y) = \frac{1}{2} y^T P^{-1} y$.

   (b) [GTHL06] For V, V^* positive definite, show that

$$\partial V(x)^T A x < 0 \text{ for all } x \in \mathbf{R}^n \quad \Longleftrightarrow \quad \partial V^*(y)^T A^T y < 0 \text{ for all } y \in \mathbf{R}^n.$$

   By the set notation $\partial f(x)^T z < 0$ we mean $y^T z < 0$ for all $y \in \partial f(x)$. Interpret the equivalence in terms of the Lyapunov functions for the system $\dot{x} = Ax$ and its dual system $\dot{y} = A^T y$.

10. *Essay question.* Many distinguished scholars argue that everything known about state space, or "modern," control theory can be reduced to successive applications of the singular value decomposition. To what extent is this view true or false? Cite specific (quantitative and qualitative) examples of this viewpoint, and discuss its merits and pitfalls. You might find the following thought experiments useful.

   - *SVD-landia.* Suppose we lived in a world where the SVD of any size matrix could be computed in constant time regardless of the dimensions. Why would you want or not want to be a resident of such a world?
   - *Not-impossiblia.* As an approximation of SVD-landia, suppose the full SVD of any matrix that fits in memory, say 10^3GB, could be computed in $1\mu s$ or less (including the time it takes to load, operate on the matrix, and store the result) in a package the size of a cellphone. How happy are you as a controls researcher?
   - *Patentville.* Suppose that every time anyone wished to solve $Ax = b$, the matrix A had to be sent, via post, to a Central Bureaucracy, where well-paid clerks would manually compute the QR decomposition and send back, via post, the matrices Q and R. How would you change your mode of operations?

11. *Convolution equation.* In this problem you will derive the continuous time convolution equation using discrete time ideas. Consider the driven system

$$\begin{cases} \dot{x}(t) = Ax(t) + v(t) \\ y(t) = Cx(t) + w(t) \\ x(0) = x_0 \end{cases}$$

   where A, C are appropriately sized real matrices and $v(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^p$ are continuous driving terms. Let $\delta \ll 1$ be a small time interval and consider the zero-order hold, forward Euler discretization of the state in the above system,

$$\frac{x_{k+1} - x_k}{\delta} = Ax_k + v_k$$
$$x(0) = x_0,$$

where for each $k = 0, 1, 2, \ldots$ we have $x(k\delta) = x_k$, $v(k\delta) = v_k$.

(a) Show that the discretization above leads to

$$x_k = (I + A\delta)^k x_0 + \sum_{i=0}^{k-1} (I + A\delta)^{k-1-i} v_i \delta, \quad k = 0, 1, 2, \ldots$$

(b) Take appropriate Riemann limits above to obtain the convolution formula,

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} v(\tau)\, d\tau.$$

You can take the following limit as given:

$$\lim_{\delta \to 0^+} (I + A\delta)^{\lfloor t/\delta \rfloor} = e^{At}.$$

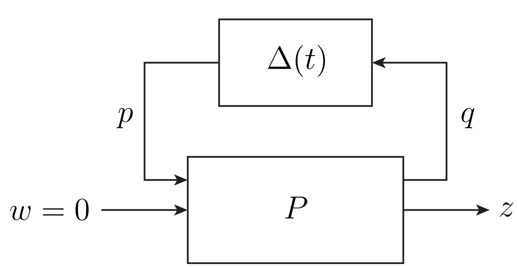
(c) Show that the output for the original continuous time system is

$$y(t) = Ce^{At} x_0 + \int_0^t Ce^{A(t-\tau)} v(\tau)\, d\tau + w(t).$$

12. *Quadratic stability margins.* [BEFB94, §5.1.2] Consider the following system,

$$\begin{cases} \dot{x} = Ax + B_p p \\ q = C_q x \\ p = \Delta(t) q, \end{cases} \tag{1}$$

where $\Delta(t) = \Delta \in \mathbf{R}^{n_p \times n_q}$ is a matrix that satisfies $\|\Delta\|_2 \leq \alpha$, and all initial conditions are zero. This can be thought of as an autonomous plant with zero input and an unknown but constant feedback perturbation as illustrated below.



(a) Show that the transfer function from p to q (without feedback perturbation) is $H_{qp}(s) = C_q(sI - A)^{-1} B_p$.

(b) Show that the condition $p = \Delta q$ and $\|\Delta\|_2 \leq \alpha$ can be rewritten as

$$p^T p \leq \alpha^2 q^T q \quad \text{for all } p, q.$$

Conclude that (x, p) must satisfy

$$\begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} \alpha^2 C_q^T C_q & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \geq 0. \tag{2}$$

(c) Use the quadratic Lyapunov function candidate $V(x) = x^T P x$ to show that the system is quadratically stable if there exists $P \succ 0$ such that

$$\begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB_p \\ B_p^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} < 0$$

for all nonzero (x, p) satisfying (2).

(d) Suppose (2) is strictly feasible. Combine the previous two parts with the lossless $\mathcal{S}$-procedure to show that the system is quadratically stable if there exists $P \succ 0$ and $\tau \geq 0$ such that

$$\begin{bmatrix} A^T P + PA + \tau\alpha^2 C_q^T C_q & PB_p \\ B_p^T P & -\tau I \end{bmatrix} \prec 0,$$

or equivalently, if there exists $\tilde{P} \succ 0$ such that

$$\begin{bmatrix} A^T \tilde{P} + \tilde{P} A + \alpha^2 C_q^T C_q & \tilde{P} B_p \\ B_p^T \tilde{P} & -I \end{bmatrix} \prec 0.$$

(e) The quadratic stability margin is the largest value $\alpha \geq 0$ for which the system is quadratically stable, which can be solved by the SDP in $\tilde{P}$, $\beta = \alpha^2$, given by

$$\begin{array}{ll} \text{maximize} & \beta \\ \text{subject to} & \tilde{P} \succ 0, \quad \beta \geq 0 \\ & \begin{bmatrix} A^T \tilde{P} + \tilde{P} A + \beta C_q^T C_q & \tilde{P} B_p \\ B_p^T \tilde{P} & -I \end{bmatrix} \prec 0. \end{array}$$

Use the bounded-real lemma (KYP) to interpret the quadratic stability margin in terms of the $\mathbf{H}_\infty$ norm of the transfer function $H_{qp}(s)$.

13. *LQR stability margins.* In this problem, we will compute guaranteed stability margins for infinite horizon LQR. Consider the state feedback system

$$\begin{cases} \dot{x} = Ax + B\Delta u \\ u = Kx, \end{cases} \tag{3}$$

where $K = -R^{-1}B^TP$, and $P \succ 0$ satisfies the algebraic Riccati equation

$$A^TP + PA - PBR^{-1}B^TP + Q = 0$$

for given matrices $Q \succ 0$, $R \succ 0$. Assume that R is diagonal. Show that the system (3) is quadratically stable with Lyapunov function $V(x) = x^TPx$ if the matrix $\Delta \in \mathbf{R}^{n_u \times n_u}$ is diagonal and $\Delta_{ii} \geq 1/2$ for all $i = 1, \ldots, n_u$. Thus, LQR is robust to 1/2 gain reduction and unbounded gain amplification in each input channel.

Hint. Use the Riccati equation to show that

$$\dot{V}(x) = x^T(PB(R^{-1} - R^{-1}\Delta^T - \Delta R^{-1})B^TP - Q)x < 0 \quad \text{for all } x \neq 0$$

if $R^{-1} - R^{-1}\Delta^T - \Delta R^{-1} \preceq 0$, and that this happens if Δ is diagonal with $\Delta_{ii} \geq 1/2$.

14. *Lyapunov equation.* Let A be a Hurwitz matrix and let $Q = Q^T \succeq 0$. Show that the Lyapunov equation $A^TP + PA + Q = 0$ has a solution given by the limit

$$P = \lim_{T\to\infty} \int_0^T e^{A^Tt}Qe^{At}\,dt.$$

Furthermore, show that if $Q \succ 0$, then $P \succ 0$ as well.

15. *Certificate of instability.* Let $\dot{x} = Ax$ be an autonomous linear system and suppose there exists a function $V(x) = x^TPx$, with $P \not\succeq 0$ such that $A^TP + PA \preceq 0$. Prove that A is not (Hurwitz) stable.

16. *Discrete Lyapunov inequality.* In this problem, we will relate the classical Lyapunov inequality $A^TP + PA \prec 0$ to the discrete time version $A_d^TPA_d - P \prec 0$. We are concerned with the autonomous (continuous time) linear system on $\mathbf{R}^n$,

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0. \end{cases} \tag{4}$$

(a) Let $x_{k+1} = A_dx_k$ be an arbitrary discrete time system on $\mathbf{R}^n$. Show that

$$\lim_{k\to\infty} x_k = 0$$

for every initial condition x_0 if there exists a quadratic Lyapunov function $V(z) = z^TPz$ such that

$$P \succ 0, \quad A_d^TPA_d - P \prec 0.$$

Hint. Show that the sequence $V_k = x_k^TPx_k$ converges to zero.

(b) Now consider the forward Euler discretization of (4),

$$\frac{x_{k+1} - x_x}{\delta} = Ax_k, \quad k = 0, 1, 2\ldots,$$

where $\delta \ll 1$ is a small time interval and $x_k = x(k\delta)$ for all $k = 0, 1, 2, \ldots$. Suppose that the forward Euler discretized system is stable, *i.e.*, with $A_d = I + A\delta$ there exists $P \succ 0$ such that

$$(I + A\delta)^T P(I + A\delta) - P \prec 0.$$

Show that $A^T P + PA \prec 0$. Thus the continuous time system (4) is stable as well.

(c) Conversely, suppose the continuous time system (4) is stable with $A^T P + PA \prec 0$ for some Lyapunov matrix $P \succ 0$. Show that there exists a small enough $\delta > 0$ such that the forward Euler discretization is stable.

17. *Worst-case analysis.* Consider the system from exercise 12,

$$\begin{cases} \dot{x} = Ax + B_p p \\ q = C_q x \\ p = \Delta(t) q, \end{cases} \tag{5}$$

with initial condition $x(0) = x_0 \in \mathbf{R}^n$, where $\Delta(t) \in \mathbf{R}^{n_p \times n_q}$ is a feedback perturbation that satisfies $\|\Delta(t)\|_2 \leq \alpha$ for all $t \geq 0$. Let $Q = Q^T \succ 0$ be a given matrix, and define a quadratic index

$$J(x_0) = \int_0^\infty x(t)^T Q x(t)\, dt,$$

which depends on the initial condition x_0. Note that J can be infinite.

(a) Suppose there exists a Lyapunov function $V(z) = z^T P z$, $P \succ 0$, with

$$\dot{V}(x) \leq -x^T Q x$$

for all (x, p) satisfying $p^T p \leq \alpha^2 x^T C_q^T C_q x$. Show that $J(x_0) \leq V(x_0)$. If we think of $J(x_0)$ as a cost associated with a trajectory of (5), this means that $V(x_0)$ is an upper bound on the worst possible cost over all instances $\|\Delta(t)\|_2 \leq \alpha$.

(b) Use the $\mathcal{S}$-procedure to show that $J(x_0) \leq V(x_0)$ if there exists P, τ such that

$$P \succ 0, \quad \tau \geq 0, \quad \begin{bmatrix} A^T P + PA + Q + \tau\alpha^2 C_q^T C_q & PB_p \\ B_p^T P & -\tau I \end{bmatrix} \preceq 0.$$

(c) For given $\alpha \geq 0$ and x_0, describe a procedure that produces a very good (*i.e.*, small) upper bound on $J(x_0)$.

18. *Discrete time bounded real lemma.* Show that the discrete time system

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k + Du_k, \quad x_0 = 0, \quad k = 0, 1, 2, \ldots$$

is nonexpansive, *i.e.*,

$$\sum_{k=0}^{T} y_k^T y_k \leq \sum_{k=0}^{T} u_k^T u_k, \qquad \text{for all } T \geq 0,$$

if there exists a matrix $P \succ 0$ such that

$$\begin{bmatrix} A^TPA - P + C^TC & A^TPB + C^TD \\ B^TPA + D^TC & B^TPB + D^TD - I \end{bmatrix} \preceq 0.$$

Hint. Use the quadratic storage function $V(x) = x^TPx$, $P = P^T \succ 0$, and impose the condition $\Delta V(x) + y^Ty - u^Tu \leq 0$ for all (x, u).

19. $\mathbf{H}_\infty$-*synthesis with single entry uncertainty.* A state space system obeys the differential equation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_u u(t) + B_w w(t) \\ z(t) &= C_z x(t) + D_{zu} u(t), \end{aligned}$$

where $x(t) \in \mathbf{R}^n$ is the state, $z(t) \in \mathbf{R}^{n_z}$ is a regulated output, $u(t) \in \mathbf{R}^{n_u}$ is a control input, and $w(t) \in \mathbf{R}^{n_w}$ is an external disturbance. Assume the initial condition $x(0) = 0$.

Every entry of the matrix $A(t)$ is constant for all t, except for a single entry at a given index (i, j), which varies (smoothly) in a known range around a nominal value, *i.e.*, $A_{ij}(t) \in [A_{ij}^{\text{nom}} - \delta, A_{ij}^{\text{nom}} + \delta]$ for some fixed $\delta > 0$ and for all t.

We would like to design a state-feedback control law $u(t) = Kx(t)$ such that

- the closed loop system is stable for all $A(t)$ in the uncertainty model
- the disturbance-to-output $\mathbf{L}_2$ gain is as small as possible

(a) Show how to model the uncertain system as a norm-bound LDI, *i.e.*, in the form

$$\begin{aligned} \dot{x}(t) &= A^{\text{nom}}x(t) + B_p p(t) + B_u u(t) + B_w w(t) \\ z(t) &= C_z x(t) + D_{zu} u(t) \\ q(t) &= C_q x(t) \\ p(t) &= \Delta(t)q(t), \quad \|\Delta(t)\| \leq 1. \end{aligned}$$

Identify the matrices A^{nom}, B_p, C_q, $\Delta(t)$, and their dimensions.

(b) Consider the closed loop system with a state-feedback control law $u(t) = Kx(t)$ in place, which has the form

$$\begin{aligned} \dot{x}(t) &= (A^{\text{nom}} + B_u K)x(t) + B_p p(t) + B_w w(t) \\ z(t) &= (C_z + D_{zu}K)x(t) \\ q(t) &= C_q x(t) \\ p(t) &= \Delta(t)q(t), \quad \|\Delta(t)\| \leq 1, \end{aligned}$$

where the last line can be replaced by the pointwise constraint $p^Tp \leq q^Tq$. Show that the disturbance-to-output $\mathbf{L}_2$ gain is bounded above by γ,

$$\sup_{\|w(\cdot)\|_2^2 \neq 0} \frac{\|z(\cdot)\|_2^2}{\|w(\cdot)\|_2^2} \leq \gamma^2,$$

if there exists a quadratic storage function $V(x) = x^T P x$, $P = P^T \succ 0$, with $\dot{V} + z^T z - \gamma^2 w^T w \le 0$ for all x, p, w and $q = C_q x$ satisfying the LDI.

(c) Use the $\mathcal{S}$-procedure to show that such a storage function exists if there exists a matrix $P = P^T \succ 0$ and a real number $\tau \ge 0$ with

$$\begin{bmatrix} \begin{pmatrix} (A^{\text{nom}} + B_u K)^T P + P(A^{\text{nom}} + B_u K) \\ +(C_z + D_{zu} K)^T (C_z + D_{zu} K) + \tau C_q^T C_q \end{pmatrix} & PB_p & PB_w \\ B_p^T P & -\tau I & 0 \\ B_w^T P & 0 & -\gamma^2 I \end{bmatrix} \preceq 0.$$

(d) Use the change of variables $Q = P^{-1}$, $Y = KQ$ to argue that the previous matrix inequality is equivalent to $Q \succ 0$, $\mu \ge 0$, and

$$\begin{bmatrix} \begin{pmatrix} A^{\text{nom}} Q + Q(A^{\text{nom}})^T + Y^T B_u^T + B_u Y \\ +\mu B_p B_p^T + B_w B_w^T \end{pmatrix} & QC_q^T & QC_z^T + Y^T D_{zu}^T \\ C_q Q & -\mu I & 0 \\ C_z Q + D_{zu} Y & 0 & -\gamma^2 I \end{bmatrix} \preceq 0.$$

Hint. You may wish to take and "un"-take several Schur complements.

(e) Show how to use the LMI from the previous part to make γ (the upper bound on the $\mathbf{L}_2$ gain from w to z) as small as possible, and how to determine the optimal state feedback gain matrix K.

(f) Implement your method for the specific data

$$A^{\text{nom}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \quad B_u = B_w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D_{zu} = 1$$

with $(i, j) = (3, 3)$ and $\delta = 1$. In other words, there is $\pm 100\%$ uncertainty in the $(3, 3)$ entry of A. Is your method numerically well conditioned? (What is the condition number of $Q^\star$?)

20. *LQR for affine LTV systems.* The dynamics for a linear time-varying discrete time system are

$$x_{t+1} = A_t x_t + B_t u_t + c_t, \quad t = 0, 1, \ldots,$$

where $A_t \in \mathbf{R}^{n \times n}$, $B_t \in \mathbf{R}^{n \times m}$ and $c_t \in \mathbf{R}^n$ are parameters that depend only on the discrete time index t. Write down the Bellman recursion for minimizing the quadratic cost index

$$J = \left(\sum_{\tau=0}^{T-1} x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right) + x_T^T Q_T x_T, \quad Q \succeq 0, \quad R \succ 0,$$

subject to the time-varying dynamics and initial condition $x_0 = z$. Show how to explicitly compute every step. *Hint.* represent the value function as

$$V_t(z) = \begin{bmatrix} z \\ 1 \end{bmatrix} \begin{bmatrix} P_t & q_t \\ q_t^T & r_t \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}, \quad t = 0, 1, \ldots, T,$$

where $P_t = P_t^T \in \mathbf{R}^{n\times n}$, $q_t \in \mathbf{R}^n$, and $r_t \in \mathbf{R}$ for all $t = 0, 1, \ldots, T$, and show how these parameters update with time.

21. *Exploiting subsystem structure.* Consider the set of linear matrix inequalities

$$P \succ 0, \quad A^T P + PA + 2\alpha P \preceq 0, \tag{6}$$

where α is a given real number, and $P = P^T$ is the variable. The matrix A has the special structure

$$A = A_0 \otimes I_m = \begin{bmatrix} (A_0)_{11} I_m & \cdots & (A_0)_{1n} I_m \\ \vdots & \ddots & \vdots \\ (A_0)_{n1} I_m & \cdots & (A_0)_{nn} I_m \end{bmatrix} \in \mathbf{R}^{nm\times nm},$$

where A_0 is a (small) $n \times n$ matrix, I_m is the $m \times m$ identity matrix, and $\otimes$ is the Kronecker product.

(a) Show that there exists a matrix $P \in \mathbf{R}^{nm\times nm}$ satisfying (6) if and only if there exists a matrix $P_0 \in \mathbf{R}^{n\times n}$ satsifying

$$P_0 \succ 0, \quad A_0^T P_0 + P_0 A_0 + 2\alpha P_0 \preceq 0. \tag{7}$$

(b) The following 2000×2000 matrix consists of four diagonal 1000×1000 blocks. Determine if it is Hurwitz stable, and if so, find its stability degree (*cf.* exercise 3).

$$A = \left[\begin{array}{ccc|ccc} -1.5 & & 0 & 1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & -1.5 & 0 & & 1 \\ \hline -1 & & 0 & 0 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & -1 & 0 & & 0 \end{array}\right]$$

22. *Gramian inequalities.* Let $A \in \mathbf{R}^{n\times n}$, $C \in \mathbf{R}^{p\times n}$ be given real matrices, and $P_0, P \in \mathbf{S}^n$ be symmetric matrices. Prove the following.

(a) *Generalized upper observability Gramian.* Suppose A is Hurwitz stable, $A^T P + PA + C^T C \preceq 0$, and $A^T P_0 + P_0 A + C^T C = 0$. Then $P_0 \preceq P$.

(b) *Generalized lower observability Gramian.* Suppose A is Hurwitz stable, $A^T P + PA + C^T C \succeq 0$, and $A^T P_0 + P_0 A + C^T C = 0$. Then $P_0 \succeq P$.

(c) *Observability.* [DP00] Suppose $A^TP+PA+C^TC = 0$, then any two of the following statements imply the third:

i. A is Hurwitz stable,

ii. (A, C) is observable,

iii. $P \succ 0$.

23. *Newton's method for ARE.* Given matrices $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{S}^n_+$, and $C \in \mathbf{S}^n$, we wish to find the unique positive definite matrix $P \in \mathbf{S}^n_{++}$ (if it exists) satisfying the algebraic Riccati equation,

$$\mathcal{R}(P) \triangleq A^TP + PA - PBP + C = 0,$$

where $\mathcal{R} : \mathbf{S}^n \to \mathbf{S}^n$ is the Riccati operator. We solve $\mathcal{R}(P) = 0$ by Newton's method, assuming (A, B) is controllable, and (A, C) is observable.

(a) Show that the (directional) derivative of $\mathcal{R}$ in the matrix direction $S \in \mathbf{S}^n$ is

$$\begin{aligned} D\mathcal{R}(P) \cdot S &\triangleq \frac{d}{dt}\big(\mathcal{R}(P + tS)\big)\Big|_{t=0} \\ &= (A - BP)^TS + S(A - BP). \end{aligned} \tag{8}$$

(b) *Initial guess.* [Sim81] Newton's method is initialized with a point P_0 for which $A - BP_0$ is Hurwitz. One way to obtain such an initial guess is to impose a minimum positive decay rate on the matrix $A-BP_0$ (see ex. 3). Given A, B, and $\alpha > 0$ we seek matrices Q and P_0 satisfying

$$Q \succ 0, \quad Q(A - BP_0)^T + (A - BP_0)Q + 2\alpha Q \preceq 0.$$

Show that by setting $Y = -P_0Q$ and eliminating Y we obtain the equivalent LMI

$$Q \succ 0, \quad \sigma \in \mathbf{R}, \quad Q(A + \alpha I)^T + (A + \alpha I)Q - \sigma BB^T \preceq 0,$$

from which we have $Y = -\frac{\sigma}{2}B^T = -P_0Q$. Finally, we can take $\sigma = 2$ by homogeneity. In fact, we can find a feasible starting point by solving two Lyapunov equations: one for Q, the other for P_0.

---

algorithm: find $P_0 \in \mathbf{S}^n$ such that $A - BP_0$ is Hurwitz
given: $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{S}^n_+$, (A, B) controllable, decay rate $0 < \alpha < \|A\|$

1. Solve for Q in $Q(A + \alpha I)^T + (A + \alpha I)Q = 2BB^T$
2. Solve for P_0 in $Q^TP_0 + P_0Q = B + B^T$

(c) *Newton step.* Suppose our current guess at timestep k is P_k. The Newton direction is given by the solution $\Delta P_{\text{nt}} \in \mathbf{S}^n$ of

$$D\mathcal{R}(P_k) \cdot \Delta P_{\text{nt}} = -\mathcal{R}(P_k).$$

Show that we can compute $\Delta P_{\rm nt}$ by solving the Lyapunov equation

$$(A - BP_k)^T \Delta P_{\rm nt} + \Delta P_{\rm nt}(A - BP_k) = -\mathcal{R}(P_k), \tag{9}$$

provided $A - BP_k$ is Hurwitz and $\mathcal{R}(P_k) \succeq 0$. Explain how to solve (9) using standard linear algebra operations like matrix multiplication and inversion.

(d) *Exact line search.* [BB98] To ensure feasibility of all iterates we will use a step length $t \in (0, 2)$ that minimizes $\|\mathcal{R}(P_k + t\Delta P_{\rm nt})\|_F^2$. Show that the optimal step length at step k is a minimizer of the quartic polynomial

$$f_k(t) = \alpha(1-t)^2 - 2\beta(1-t)t^2 + \gamma t^4, \tag{10}$$

where α, β, and γ are real numbers given by

$$\alpha = \|\mathcal{R}(P_k)\|_F^2, \quad \beta = \mathbf{Tr}(\mathcal{R}(P_k)V_k), \quad \gamma = \|V_k\|_F^2,$$

and $V_k = \Delta P_{\rm nt} B \Delta P_{\rm nt}$.

(e) *Stopping criterion.* The algorithm stops when the Newton decrement λ_k is small enough. The Newton decrement λ_k is defined in terms of the instantaneous decrease of $\|\mathcal{R}\|_F^2$ in the Newton direction,

$$-\lambda_k^2 = \frac{d}{dt}\|\mathcal{R}(P_k + t\Delta P_{\rm nt})\|_F^2\Big|_{t=0},$$

Show that $\lambda_k = \sqrt{2}\|\mathcal{R}(P_k)\|_F$.

(f) Putting these steps together, we obtain a Newton-style algorithm for solving the ARE. Implement this method on the specific matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad B = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

and compare it to Matlab's built-in method `are(A,B,C)`.

---

algorithm: Newton's method
given: $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{S}_+^n$, $C \in \mathbf{S}^n$, tolerance $\epsilon > 0$,
(A, B) controllable, (A, C) observable.

initialize:
Find P_0 such that $A - BP_0$ is Hurwitz stable.
$k := 0$

repeat:
1. *Newton step.* Solve for $\Delta P_{\rm nt}$ and λ_k
$(A - BP_k)^T \Delta P_{\rm nt} + \Delta P_{\rm nt}(A - BP_k) = -\mathcal{R}(P_k)$

$\lambda_k := \sqrt{2}\|\mathcal{R}(P_k)\|_F$

2. *Stopping criterion.* **quit** if $\lambda_k \leq \epsilon$
3. *Exact line search.*

$V_k := \Delta P_{\rm nt} B \Delta P_{\rm nt}$

$\alpha := \|\mathcal{R}(P_k)\|_F^2, \quad \beta := \mathbf{Tr}(\mathcal{R}(P_k)V_k), \quad \gamma := \|V_k\|_F^2$

Find the minimizer t_k of the quartic polynomial

$f_k(t) = \alpha(1-t)^2 - 2\beta(1-t)t^2 + \gamma t^4$

4. *Update.*

$P_{k+1} := P_k + t_k \Delta P_{\rm nt}$

$k := k+1$

References

[AG03] Farid Alizadeh and Donald Goldfarb. Second-order cone programming. *Mathematical Programming*, 95(1):3–51, 2003.

[AM14] Amir Ali Ahmadi and Anirudha Majumdar. DSOS and SDSOS optimization: LP and SOCP-based alternatives to sum of squares optimization. In *Conference on Information Sciences and Systems (CISS)*, pages 1–5, March 2014.

[BB98] Peter Benner and Ralph Byers. An exact line search method for solving generalized continuous-time algebraic Riccati equations. *IEEE Transactions on Automatic Control*, 43(1):101–107, January 1998.

[BCPT05] Erik G. Boman, Doron Chen, Ojas Parekh, and Sivan Toledo. On factor width and symmetric H-matrices. *Linear Algebra and its Applications*, 405(0):239–248, 2005.

[BEFB94] Stephen P. Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied and Numerical Mathematics*. SIAM, 1994.

[dMHBL01] Maria del Mar Hershenson, Stephen P. Boyd, and Thomas H. Lee. Optimal design of a CMOS op-amp via geometric programming. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 20(1):1–21, January 2001.

[Doy78] John C. Doyle. Guaranteed margins for LQG regulators. *IEEE Transactions on Automatic Control*, 23(4):756–757, August 1978.

[DP00] Geir E. Dullerud and Fernando Paganini. *A Course in Robust Control Theory: a Convex Approach*. Texts in Applied Mathematics. Springer, 2000.

[GTHL06] Rafal Goebel, Andrew R. Teel, Tingshu Hu, and Zongli Lin. Conjugate convex Lyapunov functions for dual linear differential inclusions. *IEEE Transactions on Automatic Control*, 51(4):661–666, April 2006.

[LVBL98] Miguel Sousa Lobo, Lieven Vandenberghe, Stephen Boyd, and Hervé Lebret. Applications of second-order cone programming. *Linear Algebra and its Applications*, 284(1–3):193–228, 1998.

[Roc70] R. Tyrrell Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

[Sim81] V. Sima. An efficient Schur method to solve the stabilizing problem. *IEEE Transactions on Automatic Control*, 26(3):724–725, June 1981.