Lecture 8. Applications

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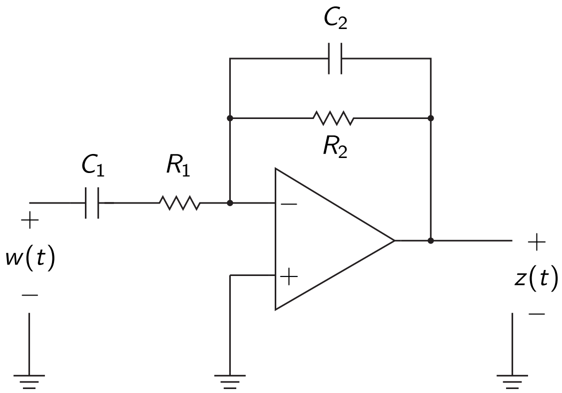
CDS270–2: Mathematical Methods in Control and System Engineering

May 18, 2015

Logistics

- hw7 due this **Wed, May 20**
  - do an easy problem or CYOA
- hw8 (design problem) will be the last homework
- hw6 solutions posted online
- reading: lmibook Ch 7

Bandpass filter



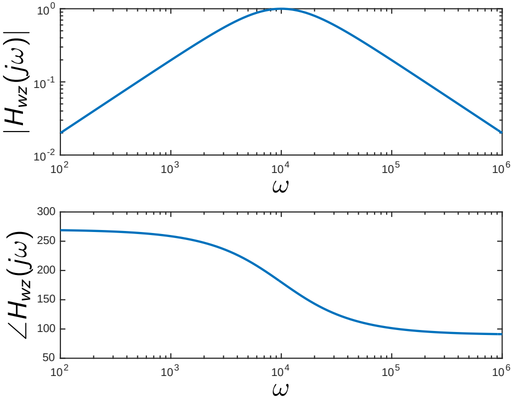
Transfer function from w to z (zero initial conditions, no clipping):

$$H_{wz}(s) = \frac{-\left(\frac{1}{sC_2}||R_2\right)}{\frac{1}{sC_1} + R_1} = \frac{-R_2 C_1 s}{(1 + R_1 C_1 s)(1 + R_2 C_2 s)}$$

Baseline design

bandpass filter:

$$R_1 = 10\text{K}, \quad R_2 = 20\text{K}, \quad C_1 = 10\text{nF}, \quad C_2 = 5\text{nF}$$



State space

put system in (minimal) state space form

$$\dot{x} = Ax + B_w w, \quad z = C_z x, \quad x(0) = 0,$$

where the matrices depend on component values,

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{R_1 R_2 C_1 C_2} & -\frac{1}{R_1 C_1} - \frac{1}{R_2 C_2} \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ \frac{1}{R_1 C_2} \end{bmatrix}, \quad C_z = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

- A is Hurwitz, (A, B_w) controllable, (A, C_z) observable
- observability Gramian: $A^T W_{\text{obs}} + W_{\text{obs}} A + C_z^T C_z = 0$

$$W_{\text{obs}} = \frac{1}{2(R_1 C_1 + R_2 C_2)} \begin{bmatrix} 1 & 0 \\ 0 & R_1 C_1 R_2 C_2 \end{bmatrix}$$

- controllability Gramian: $W_{\text{contr}} A^T + A W_{\text{contr}} + B_w B_w^T = 0$

$$W_{\text{contr}} = \frac{R_2 C_1}{2(R_1 C_1 + R_2 C_2)} \begin{bmatrix} R_2 C_1 & 0 \\ 0 & \frac{1}{R_1 C_2} \end{bmatrix}$$

Input–output norms

$\mathbf{H}_2$-norm:

$$
\begin{aligned}
\|H_{zw}\|_2^2 &= \mathbf{Tr}(B_w^T W_{\text{obs}} B_w) \\
&= \mathbf{Tr} \begin{bmatrix} 0 \\ \frac{1}{R_1 C_2} \end{bmatrix}^T \frac{1}{2(R_1 C_1 + R_2 C_2)} \begin{bmatrix} 1 & 0 \\ 0 & R_1 C_1 R_2 C_2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{R_1 C_2} \end{bmatrix} \\
&= \mathbf{Tr}(C_z W_{\text{contr}} C_z^T) \\
&= \mathbf{Tr} \begin{bmatrix} 0 & -1 \end{bmatrix} \frac{R_2 C_1}{2(R_1 C_1 + R_2 C_2)} \begin{bmatrix} R_2 C_1 & 0 \\ 0 & \frac{1}{R_1 C_2} \end{bmatrix} \begin{bmatrix} 0 & -1 \end{bmatrix}^T \\
&= \frac{R_2 C_1}{2 C_1 C_2 R_1^2 + 2 C_2^2 R_1 R_2} \\
&= 10000 \\
&\implies \|H_{zw}\|_2 = 100
\end{aligned}
$$

Input–output norms

$\mathbf{H}_\infty$-norm: solve the SDP with specific component values:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & P \succ 0, \\ & \begin{bmatrix} A^T P + PA + C_z^T C_z & PB_w \\ B_w^T P & -\gamma I \end{bmatrix} \preceq 0 \end{array}$$

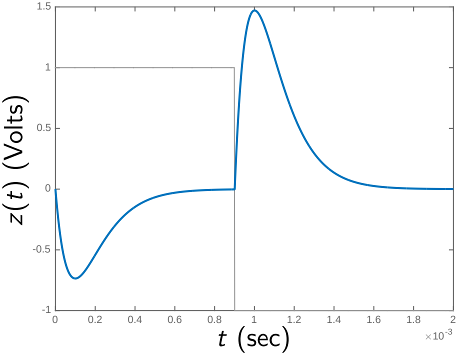
- $\gamma^\star = \|H_{wz}\|_\infty^2 = \|C_z(sI - A)^{-1} B_w\|_\infty^2 = 1.00$
- can also read $\|H_{wz}\|_\infty$ directly from Bode magnitude plot

$$\|H_{wz}\|_\infty = \sup_\omega \sigma_{\max}(H_{wz}(j\omega)) = 1.00$$

Peak gain

question. is it possible for op-amp to clip even though $\|H_{wz}\|_\infty = 1$?

$$\|H_{wz}\|_\infty = \sup_{w\neq 0} \frac{\|z\|_2}{\|w\|_2} = 1, \quad \|H_{wz}\|_{\text{pk-gain}} = \sup_{w\neq 0} \frac{\|z\|_\infty}{\|w\|_\infty} \geq 1.4$$



answer. yes (depending on supply limits, scale & apply signal above)

Component variations

Suppose each component can vary by $\pm 10\%$. Is stability guaranteed?

polytopic LDI. model the state space matrices

$$\begin{bmatrix} A & B_w \end{bmatrix} \in \mathbf{conv}\left\{ \begin{bmatrix} A_1 & (B_w)_1 \end{bmatrix}, \ldots, \begin{bmatrix} A_L & (B_w)_L \end{bmatrix} \right\}$$

autonomous stability certificate. find a joint quadratic Lyapunov function $V(x) = x^T P x$, $P \succ 0$, with

$$A_i^T P + P A_i \prec 0, \quad i = 1, \ldots, L,$$

where the A_i are formed by all combinations of parameter variations, *i.e.*,

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{R_1 R_2 C_1 C_2} & -\frac{1}{R_1 C_1} - \frac{1}{R_2 C_2} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{0.9 R_1 R_2 C_1 C_2} & -\frac{1}{0.9 R_1 C_1} - \frac{1}{R_2 C_2} \end{bmatrix}$$

$$\vdots$$

Guaranteed bounds on output peak for initial condition

Suppose $\mathcal{E} = \{x \mid x^T P x \leq 1\}$ is an invariant ellipsoid containing the initial condition $x(0)$, then

$$\begin{aligned} z(t)^T z(t) &\leq \sup_{x \in \mathcal{E}} x^T C_z^T C_z x \\ &= \lambda_{\max}(P^{-1/2} C_z^T C_z P^{-1/2}) \\ &\leq \delta, \end{aligned}$$

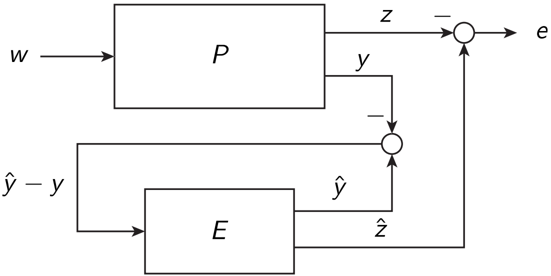
provided there is a matrix $P = P^T$ satisfying

$$P \succ 0, \quad x(0)^T P x(0) \leq 1, \quad \begin{bmatrix} P & (C_z)_i^T \\ (C_z)_i & \delta I \end{bmatrix} \succeq 0, \quad A_i^T P + P A_i \preceq 0.$$

for input–output properties, select $x(0)$ according to $(B_w)_i$.

State estimation

observer design. given plant P driven by w, find an observer gain L to minimize some norm of the w-to-e transfer function, $\|H_{ew}\|$.



plant model.

$$\dot{x} = Ax + B_w w$$
$$y = C_y x + D_{yw} w$$
$$z = C_z x$$

estimator model.

$$\dot{\hat{x}} = A\hat{x} + L(\hat{y} - y)$$
$$\hat{y} = C_y \hat{x}$$
$$\hat{z} = C_z \hat{x}, \quad e = \hat{z} - z$$

Luenberger-style architecture

plant model.

$$
\begin{aligned}
\dot{x} &= Ax + B_w w \\
y &= C_y x + D_{yw} w \\
z &= C_z x
\end{aligned}
$$

estimator model.

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\dot{\hat{x}} &= A\hat{x} + L(\hat{y} - y) \\
\hat{y} &= C_y \hat{x} \\
\hat{z} &= C_z \hat{x}, \quad e = \hat{z} - z
\end{aligned}
$$

- estimator attempts to replicate plant dynamics without w
- exogenous input w (*e.g.*, noise) is not directly accessible to estimator
- C_y determines which parts of the state can be measured
- C_z controls performance index (*e.g.*, makes units of x comparable),

$$
\|H_{ew}\| = \|C_z(\hat{x} - x)\|
$$

example. $C_z = I$, $D_{yw} = I$ with $\|\cdot\|$ given by $\mathbf{H}_2$-norm is a Kalman filter

Closed loop system

Substitute $\hat{y} = C\hat{x}$ and $y = Cx + D_{yw}w$ to obtain

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + LC_y(\hat{x} - x) - LD_{yw}w \\ \dot{x} &= Ax + B_w w,\end{aligned}$$

and write as an augmented linear system,

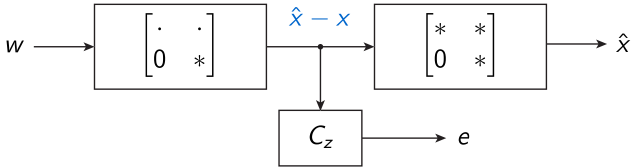
$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{x}} - \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} A & LC_y \\ 0 & A + LC_y \end{bmatrix}}_{A_{\text{cl}}} \begin{bmatrix} \hat{x} \\ \hat{x} - x \end{bmatrix} + \underbrace{\begin{bmatrix} -LD_{yw} \\ -B_w - LD_{yw} \end{bmatrix}}_{B_{\text{cl}}} w$$

$$e = \underbrace{\begin{bmatrix} 0 & C_z \end{bmatrix}}_{C_{\text{cl}}} \begin{bmatrix} \hat{x} \\ \hat{x} - x \end{bmatrix}.$$

We aim to minimize the norm of the w-to-e transfer function,

$$\|H_{ew}\| = \|C_{\text{cl}}(sI - A_{\text{cl}})^{-1}B_{\text{cl}}\|.$$

Decoupling



decomposition. the second states are driving the first states, so we do not need the entire dynamics to compute $\|H_{ew}\|$.

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{x}} - \dot{x} \end{bmatrix} = \begin{bmatrix} A & LC_y \\ 0 & A + LC_y \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x} - x \end{bmatrix} + \begin{bmatrix} -LD_{yw} \\ -B_w - LD_{yw} \end{bmatrix} w, \quad e = \begin{bmatrix} 0 & C_z \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x} - x \end{bmatrix}$$

$$\Downarrow$$

$$(\dot{\hat{x}} - \dot{x}) = (A + LC_y)(\hat{x} - x) + (-B_w - LD_{yw})w, \quad e = C_z(\hat{x} - x)$$

$$\Downarrow$$

$$\|H_{ew}\| = \|C_z(sI - (A + LC_y))^{-1}(-B_w - LD_{yw})\|$$

$\mathbf{H}_2$ case

The norm $\|H_{ew}\|_2$ is the square root of the optimal value of

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\left((B_w + LD_{yw})^T P (B_w + LD_{yw})\right) \\ \text{subject to} & P \succeq 0 \\ & (A + LC_y)^T P + P(A + LC_y) + C_z^T C_z \preceq 0 \end{array}$$

- convex program in $P = P^T$ and $W = PL$ (assuming $D_{yw} D_{yw}^T = I$ and $B_w D_{yw}^T = 0$)

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(B_w^T P B_w) + \mathbf{Tr}(W^T P^{-1} W) \\ \text{subject to} & P \succeq 0 \\ & A^T P + PA + C_z^T C_z + C_y^T W^T + W C_y \preceq 0 \end{array}$$

- optimal estimator gain is $L^\star = (P^\star)^{-1} W^\star$, and is independent of C_z

alternate solution. set $L = -QC_y^T$ in the solution Q of the algebraic Riccati equation

$$QA^T + AQ + B_w B_w^T - QC_y^T C_y Q = 0.$$

$\mathbf{H}_\infty$ case

The norm $\|H_{ew}\|_\infty$ is the square root of the optimal value of

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & P \succ 0 \\ & \begin{bmatrix} (A+LC_y)^T P + P(A+LC_y) + C_z^T C_z & P(B_w + LD_{yw}) \\ (B_w + LD_{yw})^T P & -\gamma I \end{bmatrix} \preceq 0 \end{array}$$

- dissipation condition: $V = x^T P x$, $P \succ 0$, $\dot{V} + e^T e - \gamma w^T w \leq 0$
- convex program in $P = P^T$, γ, and $W = PL$

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & P \succ 0 \\ & \begin{bmatrix} \begin{pmatrix} A^T P + PA + C_z^T C_z \\ + C_y^T W^T + W C_y \end{pmatrix} & PB_w + WD_{yw} \\ B_w^T P + D_{yw}^T W^T & -\gamma I \end{bmatrix} \preceq 0 \end{array}$$

- optimal estimator gain is $L^\star = (P^\star)^{-1} W^\star$

Example: $\mathbf{H}_2$ vs $\mathbf{H}_\infty$ estimator

- nominal plant model (DC motor inspired)

$$A^{\text{nom}} = \begin{bmatrix} -0.1 & 1 & 0 \\ 0 & -1.1 & 1 \\ 0 & -1 & -1.1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

- estimator parameters

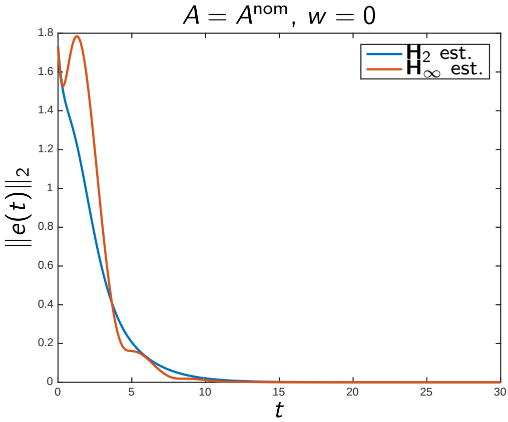
$$C_z = I_3, \quad D_{yw} = 1$$

- we will track estimator performance with three noise types

$$w(t) = 0, \quad w(t) = 1, \quad w(t) = \cos(2\pi t)$$

- would like estimator to be robust with respect to (polytopic) model perturbations.

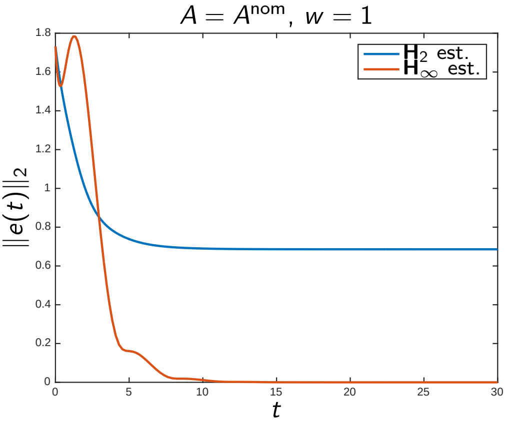
Example: $\mathbf{H}_2$ vs $\mathbf{H}_\infty$ estimator



$$L^\star_{(\mathbf{H}_2)} = (-0.31, -0.08, -0.02)$$

$$L^\star_{(\mathbf{H}_\infty)} = (0.00, 0.00, -1.00)$$

Example: $\mathbf{H}_2$ vs $\mathbf{H}_\infty$ estimator



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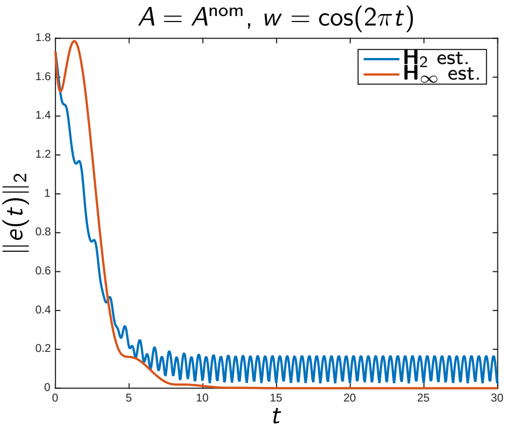

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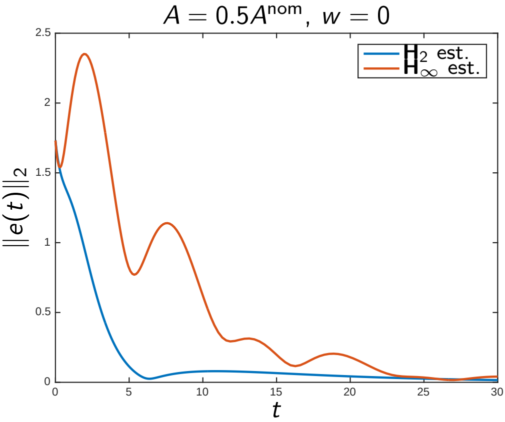

$$L^{\star}_{(\mathbf{H}_2)} = (-0.31, -0.08, -0.02)$$
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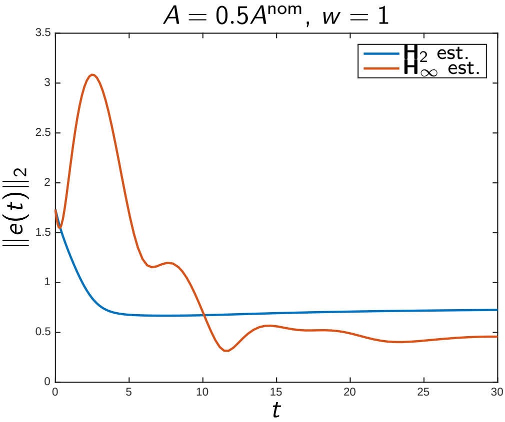

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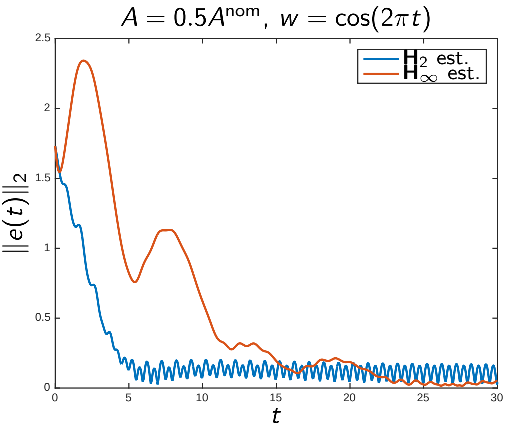

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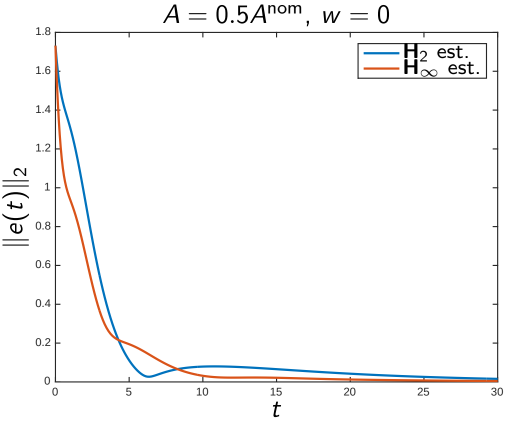
Example: robust $\mathbf{H}_2$ vs robust $\mathbf{H}_\infty$ estimator

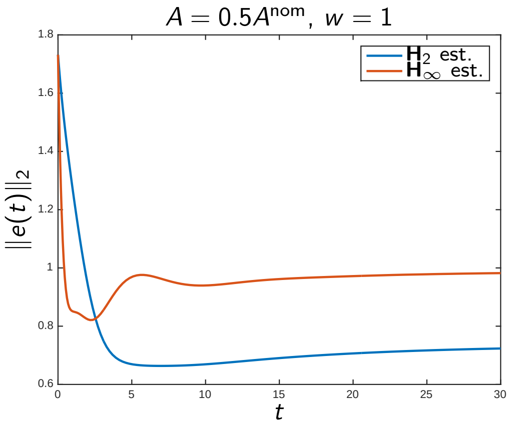


redesigned estimators, polytopic model $A \in \text{conv}\{0.25A^{\text{nom}}, 1.75A^{\text{nom}}\}$

$$L^\star_{(\mathbf{H}_2)} = (-0.31, -0.08, -0.03)$$

$$L^\star_{(\mathbf{H}_\infty)} = (-1.59, 0.20, -1.04)$$

Example: robust $\mathbf{H}_2$ vs robust $\mathbf{H}_\infty$ estimator



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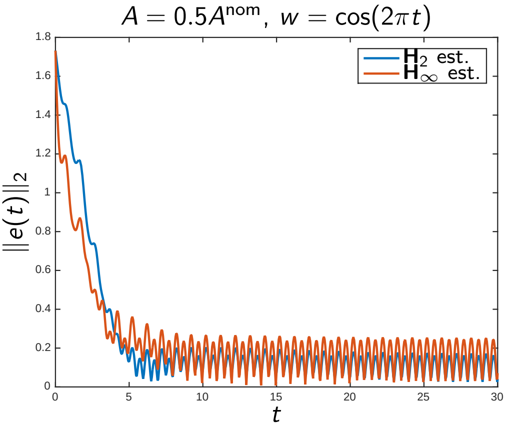

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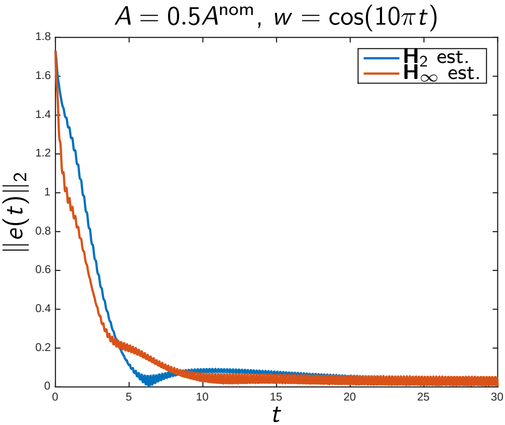

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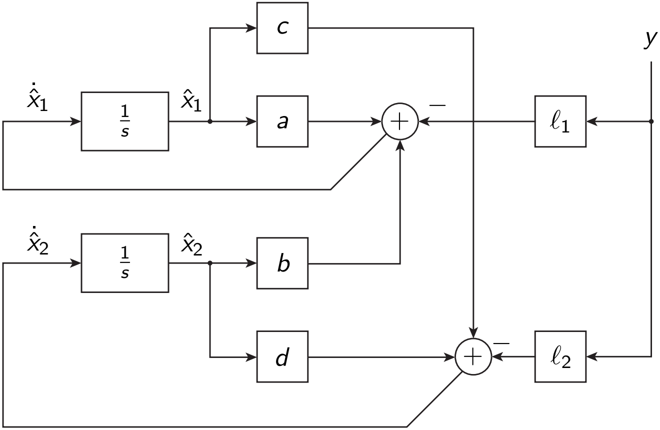
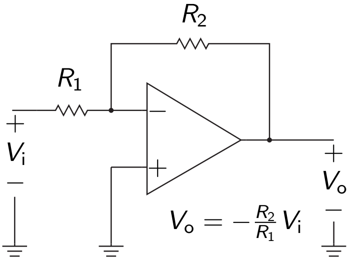
Implementation of Luenberger observer



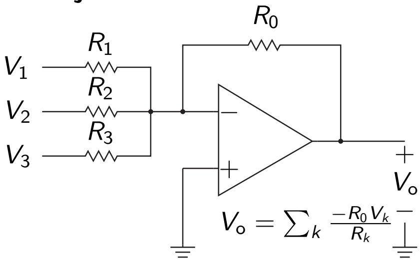
$$\dot{\hat{x}} = (A + LC_y)\hat{x} - Ly, \quad A + LC_y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$$

Analog computing elements

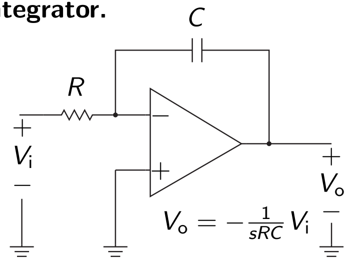
gain-inverter.



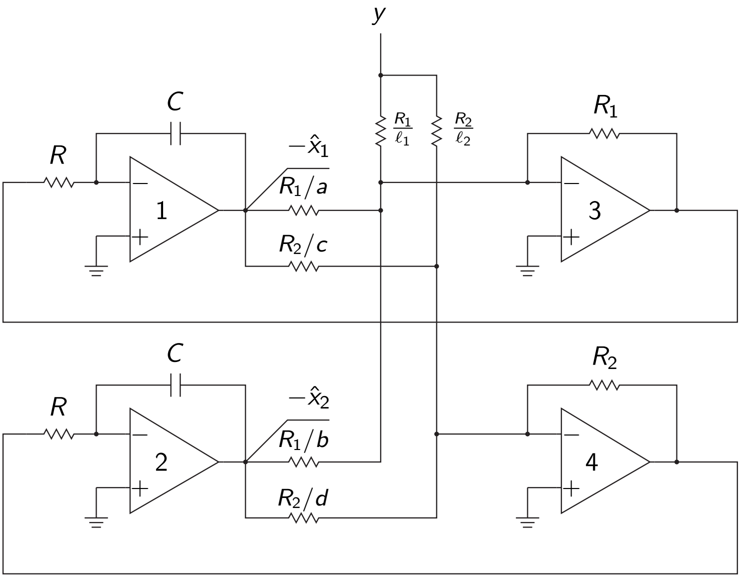
sum junction.



integrator.



Luenberger observer analog computer



Notes

- op-amps 1 and 2 are integrators, 3 and 4 are sum junctions
- R and C chosen so the time constant is $RC = 1$ second
- R_1, R_2 arbitrary, chosen so
    - all internal signals stay within supply voltage bounds
    - internal signal-to-noise ratios are not too small
    - gain ratios specified by constants a, b, c, d
- y is external (voltage) signal
- $-\hat{x}_1$ and $-\hat{x}_2$ are negative (voltage) state estimates