Collaborative System Identification via Parameter Consensus

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***Abstract*— Classical schemes in system identification and adaptive control often rely on persistence of excitation to guarantee parameter convergence, which may be difficult to achieve with a single agent and a single input. Inspired by consensus systems, we extend classical parameter adaptation to the multi agent setting by combining an adaptive gradient law with consensus dynamics. The gradient law represents the main learning signal, while consensus dynamics attract each agent's parameter estimates toward those of its neighbors. We show that the resulting decentralized online parameter estimator can be used to identify the true parameters of all agents, even if no single agent employs a persistently exciting input.**

I. Introduction

We envision collaborative system identification applications where identical intelligent agents can communicate with each other, and are tasked with reaching consensus on some set of common parameters. While we are motivated by the case where these parameters specify the continuous dynamics for a nominal model class, of which every agent in the system is an instance, our information sharing framework readily applies to more general collaborative filtering and estimation schemes. For example, the parameters can refer to a static (or slowly changing) global state that each agent in the network can only partially observe. Either way, the parameters are determined in a decentralized, adaptive way through an online scheme that integrates local measurements with communicated information.

Classical (single agent) system identification algorithms determine model parameters by probing the system with an *a priori* selected input and observing the output. If the input is "exciting" enough to stimulate all the relevant internal dynamical modes, the model parameters can be backed out by adaptation. Otherwise, most algorithms can only ascertain the parameters to the extent that they replicate the observed input-output relationship. Designing an input that guarantees parameter convergence is difficult, because the persistence of excitation (PE) conditions that must be checked often require solving for the full system trajectory.

If we can replicate the system into an ensemble of identical systems and probe each one with a different input, can the parameter estimates converge to their true values under more relaxed conditions than with just one test system? In this work, we answer this question in the affirmative, provided that a collective persistence of excitation condition holds. The condition ensures that a minimal "overall" level of input excitation is present within the communication network.

As a main ingredient of our collaborative identification scheme, we develop a parameter estimator based on a combination of linear consensus and local system identification. Consensus, agreement, and flocking have been widely studied in computer science and dynamical systems, see [1], [2] and references therein for a good introduction. The closest works to ours, [3], [4], solve a sensor fusion problem in robotic coverage applications and characterize the role of persistently excited agents as knowledge leaders in the network. More recently, similar gradient-based schemes have been analyzed with noise in [5] and sampled data in [6].

We expand upon classical parameter convergence results (*e.g.*, [7], [8], [9]) by generalizing to the networked communication case. Our main contribution is a notion of *collective* persistence of excitation that takes advantage of the information shared between agents. As a special example of the condition, we show that parameter estimates can be made to converge to their true values even if no single agent uses a persistently exciting input.

The paper is organized as follows. In §II, we review linear consensus and set up the collaborative system identification problem. The main result, Theorem 1, summarizes the collective PE condition. In §III, we motivate the estimator dynamics and show how collective PE translates to networked linear dynamical systems. For easier digestion, we prove the scalar version of Theorem 1 in §IV, and give vector modifications in the appendix. A numerical example is given in §V, and remarks in §VI conclude the paper.

II. Distributed Consensus

A. Preliminaries

An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a finite set of n vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$ together with a set of m edges $\mathcal{E} = \{e_1, \ldots, e_m\}$. We sometimes write i for v_i. An edge e_k is an unordered pair of vertices $\{v_i, v_j\} \subseteq \mathcal{V}$. The adjacency matrix of $\mathcal{G}$ is a matrix $A = [a_{ij}] \in \mathbf{R}^{n\times n}$ with entries

$$a_{ij} = \begin{cases} +1, & \text{if } \{v_i, v_j\} \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases}$$

The adjacency matrix is symmetric ($A = A^T$) for undirected graphs. The neighborhood of a vertex v_i consists of the set of adjacent vertex indices $\mathcal{N}_i = \{j \mid \{v_i, v_j\} \in \mathcal{E}\}$. The degree of v_i, written $\deg(v_i)$, is the number of neighbors $|\mathcal{N}_i|$ of that vertex, and the degree matrix is the diagonal matrix $D = \mathrm{diag}(\deg(v_1), \ldots, \deg(v_n)) \in \mathbf{R}^{n\times n}$.

The graph Laplacian $L = L^T \in \mathbf{R}^{n\times n}$ is defined as

$$L = D - A.$$

The Laplacian matrix is positive semidefinite, which we write as $L \succeq 0$, whenever $\mathcal{G}$ is connected. This follows from, *e.g.*,

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the fact that L is (weakly) diagonally dominant with strictly positive entries on the diagonal. In general, we write $A \succeq B$ to mean $A - B \succeq 0$ in the matrix sense.

A key property of connected graphs is that all eigenvalues $\lambda_1, \ldots, \lambda_n$ of L are strictly positive except for the smallest, which is zero. We order the eigenvalues of L as

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n.$$

The second smallest eigenvalue λ_2 is the known as the algebraic connectivity of $\mathcal{G}$. The (column) eigenvector $\mathbf{1} = (1, \ldots, 1) \in \mathbf{R}^n$ corresponds to the zero eigenvalue subspace. In particular,

$$L\mathbf{1} = 0, \quad \mathbf{1}^T L = 0.$$

B. Parameter Estimator Dynamics

An ensemble of n agents has communication topology $\mathcal{G}$: each vertex v_i is an *agent* and each edge $e_k = \{v_i, v_j\}$ is an allowed (bidirectional) *communication link* between agents i and j. At any time $t \geq 0$, agent i can measure a surrogate state time series $x_i(t) \in \mathbf{R}^q$ and a real-valued output $y_i(t) \in \mathbf{R}$. The surrogate state $x_i(t)$ can be, for example, a filtered version of the agent's true dynamical state. We model the output $y_i(t)$ as a linear combination of parameters,

$$y_i(t) = \theta^T \phi(x_i(t)), \tag{1}$$

where $\phi : \mathbf{R}^q \to \mathbf{R}^p$ is a known regressor and $\theta \in \mathbf{R}^p$ is a vector of fixed but unknown coefficients.

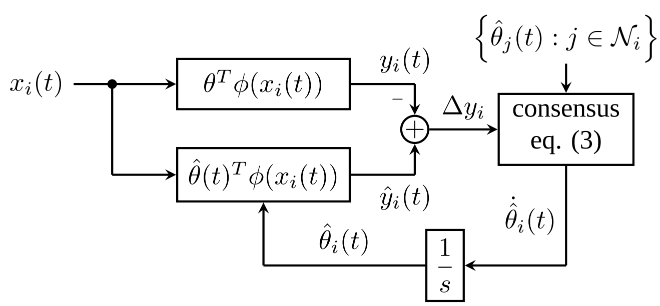


Fig. 1. Each agent in the network implements the estimator dynamics (3).

In order to determine the parameter vector θ, each agent i has an estimate $\hat{\theta}_i(t)$ of θ made from local measurements and any information communicated by the agent's neighbors. The agent generates a local prediction of the output,

$$\hat{y}_i(t) = \hat{\theta}_i(t)^T \phi(x_i(t)), \tag{2}$$

and attempts to decrease its output prediction error $\Delta y_i = \hat{y}_i - y_i$ by modifying $\hat{\theta}_i$ with time. For brevity, define $\phi_i(t) = \phi(x_i(t))$. We consider the combined estimator dynamics

$$\begin{aligned} \frac{d}{dt}\hat{\theta}_i = &-\gamma \phi_i(t)(\hat{y}_i - y_i) \\ &+ \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{\theta}_j - \hat{\theta}_i), \quad i = 1, \ldots, n. \end{aligned} \tag{3}$$

With the first term of (3) we seek to reduce the local output prediction error. The constant estimation gain $\gamma > 0$ controls the local information fusion rate. Notice that this term is linear time-varying with the parameter estimates $\hat{\theta}_i$, which can be seen by substituting (2) into (3). The second term, a sum over the neighbors $\mathcal{N}_i$, presents a mechanism for global parameter consensus by ensuring that $\hat{\theta}_i$ does not stray too far from any neighboring $\hat{\theta}_j$, where $j \in \mathcal{N}_i$. In control theory terms, these dynamics describe a linear consensus controller driven by the learning signals $-\gamma\phi_i(t)(\hat{y}_i - y)$.

The main result of this paper is that the identified parameters governed by dynamics (3) asymptotically achieve consensus $\hat{\theta}_1 = \cdots = \hat{\theta}_n$ with all signals remaining bounded. In addition, if a collective persistence of excitation condition is met, the parameter errors $\Delta\theta_i = \hat{\theta}_i - \theta$ converge to zero. The main result is summarized in the following theorem.

Theorem 1. *Suppose that $\mathcal{G}$ is connected and each regressor $\phi_i(t) = \phi(x_i(t))$ remains bounded with bounded first derivative for all $i = 1, \ldots, n$. Then the dynamics* (3) *exhibit*

1) bounded internal signals: *$\hat{\theta}_i(t)$ and $\hat{y}_i(t)$ are bounded for all $i = 1, \ldots, n$ and for all $t \geq 0$,*
2) asymptotic zero prediction error: *$\Delta y_i(t) = \hat{y}_i(t) - y_i(t) \to 0$ as $t \to \infty$ for all $i = 1, \ldots, n$.*
3) asymptotic parameter consensus: *$\hat{\theta}_j(t) - \hat{\theta}_i(t) \to 0$ as $t \to \infty$ for all $i, j = 1, \ldots, n$.*

If in addition there exist positive real numbers $m_1, m_2 > 0$ such that for all $t_0 \geq 0$ and $t > t_0$ the matrix inequality

$$m_2 I \succeq \frac{1}{t - t_0} \int_{t_0}^{t} \sum_{i=1}^{n} \phi_i(\tau)\phi_i(\tau)^T \, d\tau \succeq m_1 I \tag{4}$$

holds, then we also have

4) asymptotic parameter convergence: *the parameter errors $\Delta\theta_i(t) = \hat{\theta}_i(t) - \theta \to 0$ as $t \to \infty$ for all $i = 1, \ldots, n$.*

The condition (4) encodes a notion of *collective* persistence of excitation (PE). For the trivial network with a single agent ($n = 1$), collective PE reduces to PE of a single regressor

$$m_2 I \succeq \frac{1}{t - t_0} \int_{t_0}^{t} \phi_1(\tau)\phi_1(\tau)^T \, d\tau \succeq m_1 I, \tag{5}$$

which is sufficient to obtain parameter convergence in traditional system identification where parameter consensus plays no explicit role [7], [8], [9]. From linearity of the integral in condition (4), collective PE occurs in an ensemble $\{\phi_1, \ldots, \phi_n\}$ of regressors if, for example, any of the following types of excitation take place:

- *Enlightened*: a few ϕ_i are persistently exciting,
- *Total*: every ϕ_i is persistently exciting,
- *Intermittent*: there exists an unbounded sequence of times $t_1, t_2, \ldots$ such that some ϕ_i obeys (5) in each interval $[t_k, t_{k+1}]$,
- *Collaborative*: none of the ϕ_i is persistently exciting, but condition (4) still holds.

In the first two cases, distinguished agents have the role of a knowledge leader in the network (*cf.* [4]). The last two reveal that parameter convergence can still occur even if no single agent can claim leadership over all parameters, because the information shared through consensus reconciles any PE deficiency with other agents. This point is explored further in §V.

The hypothesis of Theorem 1 can also be rephrased with conditions on the regressor function itself, *e.g.*, $\phi : \mathbf{R}^q \to \mathbf{R}^p$ uniformly continuous in x. We prove Theorem 1 for real θ in §IV. The proof for vector $\theta \in \mathbf{R}^p$ ($p > 1$) is relegated to the appendix.

III. System Identification

A. Instantaneous Objective Minimization

We reinterpret the dynamics (3) as instantaneous minimization of a particular cost function. The dynamics arise from two main desires for the network as a whole. First, all local estimates $\hat{\theta}_i$ should converge to the same value as $t \to \infty$, and second, the value to which the local estimates converge should be the true θ. Define at each time $t \geq 0$ an instantaneous quadratic cost $J : \mathbf{R}^p \times \cdots \times \mathbf{R}^p \to \mathbf{R}$,

$$J(\hat{\theta}_1(t), \ldots, \hat{\theta}_n(t)) = \sum_{i=1}^{n} \gamma J_i(\hat{\theta}_i(t)) + \sum_{\{v_i,v_j\} \in \mathcal{E}} \frac{a_{ij}}{2} \big\|\hat{\theta}_j(t) - \hat{\theta}_i(t)\big\|_2^2, \quad (6)$$

with variables $\hat{\theta}_i(t) \in \mathbf{R}^p$ and local prediction costs $J_i : \mathbf{R}^p \to \mathbf{R}$ for all $i = 1, \ldots, n$. The dynamics (3) can be recovered from the gradient flow

$$\frac{d}{dt}\hat{\theta}_i = -\frac{\partial J}{\partial \hat{\theta}_i}, \quad i = 1, \ldots, n, \quad (7)$$

with quadratic local prediction costs

$$J_i(\hat{\theta}_i(t)) \triangleq \frac{1}{2}\big(\hat{y}_i(t) - y_i(t)\big)^2, \quad i = 1, \ldots, n,$$

where $\hat{y}_i(t)$ is given in terms of $\hat{\theta}_i(t)$ by the local prediction equation (2), and $y_i(t)$ comes from an online measurement. The learning rate $\gamma > 0$ trades off instantaneous prediction error with total parameter disagreement.

We present quadratic J_i for simplicity, though it is straightforward to devise schemes robust to process noise, communication noise, unmodeled dynamics, and parameter drift, see [10], [11], [12]. For example, if instead we use

$$J_i(\hat{\theta}_i(t)) \triangleq \int_0^t \big(\hat{\theta}_i(t)^T \phi(x_i(\tau)) - y_i(\tau)\big)^2 \, d\tau$$

in the cost (6), our collaborative identification scheme becomes more robust to measurement noise. If the prediction costs are zero ($J_i = 0$) and $p = 1$, the flow (7) reduces to gradient flow on the quadratic disagreement function

$$J(\hat{\theta}_1, \ldots, \hat{\theta}_n) = \sum_{\{v_i,v_j\} \in \mathcal{E}} \frac{a_{ij}}{2}\big(\hat{\theta}_j - \hat{\theta}_i\big)^2,$$

which is the classical linear consensus flow $\dot{\hat{\theta}} = -L\hat{\theta}$, see [1], [2]. We can also reformulate the cost (6) as a constrained objective, rather than a quadratically penalized objective, to obtain second order dynamics (PI control) with parameter consensus [13], [14].

B. Dynamical System Identification and Sufficient Richness

The persistence of excitation condition (4) is awkward to verify for generic dynamical systems even in the classical single agent setting ($n = 1$). For linear dynamical systems, it can be shown that a sinusoidal input with enough independent frequency components, *i.e.*, an input that is *sufficiently rich*, will generate the persistence of excitation necessary for parameter convergence [11], [15].

We now demonstrate how sufficient richness translates to the collaborative multi agent setting by example. Suppose the nominal system to be identified is the (stable) single input linear system

$$\dot{x}(t) = ax(t) + bu(t),$$

where $a < 0$ and b are constant (unknown) parameters. With our goal to determine a and b, we instantiate an ensemble of identical systems whose communication structure is organized by the graph $\mathcal{G}$. Each agent chooses their own input $u_i(t) \in \mathbf{R}$ and observes the resulting state $x_i(t) \in \mathbf{R}$ for all $i = 1, \ldots, n$. The dynamics are

$$\dot{x}_i(t) = ax_i(t) + bu_i(t), \quad i = 1, \ldots, n. \quad (8)$$

Note that the dynamics (8) are of the nominal form (1), where $y_i(t) \in \mathbf{R}$ is the state derivative $\dot{x}_i(t)$, the regressor is $\phi_i(t) = (x_i(t), u_i(t)) \in \mathbf{R}^2$, and the unknown parameter vector is $\theta = (a, b) \in \mathbf{R}^2$. In practice, the time derivative $\dot{x}_i(t)$ is not a signal available for measurement, so we often (linearly) filter both sides of (8) and redefine surrogate outputs and regressors by their filtered versions.

In the classical setting ($n = 1$), condition (4) is satisfied if we choose $u(t) = \sin(\omega t)$ with $\omega \neq 0$, because

$$\frac{1}{t - t_0}\int_{t_0}^{t} \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix} \begin{bmatrix} x(\tau) & u(\tau) \end{bmatrix} d\tau$$

eventually has bounded positive eigenvalues. Note that the choice of $u(t)$ determines $x(t)$, and hence the value of the integral above. In the multi agent setting ($n > 1$), there is considerably more design freedom in choosing the inputs $u_i(t)$ to obtain desired parameter convergence dynamics while maintaining collective PE, and hence a guarantee of parameter convergence.

For example, it suffices that $u_i(t) = \sin(\omega t)$ for some $i \in \{1, \ldots, n\}$, while the rest of the $u_j(t)$, for $j \neq i$, are arbitrary. The distinguished agent i can be thought of as *enlightened* to the true dynamics of the system because that agent is probed with a known sufficiently rich input. Parameter consensus then ensures that all other agents reach the same conclusion about the values of a and b as the enlightened agent i. Moreover, if all agents are enlightened, as is the case in *total* excitation, the designer of the collaborative identification system can trade off parameter dynamics (time) against the number of agents (space).

IV. Convergence of Parameter Dynamics

As stated earlier, we will prove Theorem 1 for simultaneous identification and consensus on a single real-valued parameter $\theta \in \mathbf{R}^p$, $p = 1$. The case $p > 1$ is essentially the same with more involved bookkeeping. The proof modifications for $p > 1$ are given in the appendix.

The proof relies on a standard result of analysis that a function which has a finite limit and uniformly continuous derivative has derivative that converges to zero [16]. Known as Barbalat's lemma, restated below in its integral form, it is central to proving Theorem 1, parts 1–3.

Lemma (Barbalat)**.** *Let $f : [0, \infty) \to \mathbf{R}$ be a uniformly continuous function and suppose that $\lim_{t\to\infty} \int_0^t f(\tau)\, d\tau$ exists and is finite. Then $f(t) \to 0$ as $t \to \infty$.*

To obtain parameter convergence in Theorem 1, part 4, we will use the persistence of excitation condition from [7], which gives necessary and sufficient conditions for the uniform asymptotic stability of a time-varying autonomous system. It says that the origin is the unique stable equilibrium of $\dot{x} = -P(t)x$, if the matrix $-P(t) \in \mathbf{R}^{n\times n}$ is stable, on average, in any direction in $\mathbf{R}^n$. The condition below can be expressed in many ways, and we direct the reader to the classical references [7], [8], [9], [11] for additional insight.

Theorem 2 (Morgan and Narendra 1977)**.** *Suppose $P(t)$ is a symmetric positive semidefinite matrix of bounded piecewise continuous functions. Then the equation $\dot{x} = -P(t)x$ is uniformly asymptotically stable if and only if there are real numbers $a > 0$ and b such that for all $t_0 \geq 0$ and $t \geq t_0$,*

$$\int_{t_0}^{t} w^T P(\tau) w \, d\tau \geq a(t - t_0) + b$$

for all fixed unit vectors w.

Proof of Theorem 1 for $p = 1$. Stack the real components $\hat{\theta}_i$ into a column vector $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_n) \in \mathbf{R}^n$ and let the parameter error be $\Delta\theta = \hat{\theta} - \theta\mathbf{1} \in \mathbf{R}^n$. In view of the estimator definitions in (2), each agent's individual learning signal depends on $\Delta\theta$ via

$$-\gamma\phi(x_i)(\hat{y}_i - y_i) = -\gamma\phi(x_i)^2 \Delta\theta_i.$$

Putting these together, the individual dynamics (3) can be aggregated in matrix form as

$$\frac{d}{dt}\Delta\theta = -L\Delta\theta - \gamma\Phi\Delta\theta \tag{9}$$

where $\Phi(t) = \mathrm{diag}\big(\phi_1^2(t), \ldots, \phi_n^2(t)\big) \in \mathbf{R}^{n\times n}$, and we used the identity $L\Delta\theta = L(\hat{\theta} - \theta\mathbf{1}) = L\hat{\theta}$.

Consider the candidate Lyapunov function

$$V(\Delta\theta) = \frac{1}{2}\Delta\theta^T\Delta\theta.$$

The time derivative of V along solution trajectories of (9) is

$$\begin{aligned}\dot{V}(\Delta\theta) &= \frac{1}{2}\left(\Big(\frac{d}{dt}\Delta\theta\Big)^T\Delta\theta + \Delta\theta^T\Big(\frac{d}{dt}\Delta\theta\Big)\right)\\ &= \frac{1}{2}\Big(\big(-L\Delta\theta - \gamma\Phi\Delta\theta\big)^T\Delta\theta\\ &\qquad + \Delta\theta^T\big(-L\Delta\theta - \gamma\Phi\Delta\theta\big)\Big)\\ &= -\Delta\theta^T L\Delta\theta - \gamma\Delta\theta^T\Phi\Delta\theta \qquad (10)\\ &\leq 0,\end{aligned}$$

where the inequality follows from the positive semidefiniteness of L and $\Phi(t)$ and from the learning rate assumption $\gamma > 0$.

Since V is bounded below ($V \geq 0$) and nonincreasing ($\dot{V} \leq 0$), it converges to a limit as $t \to \infty$. Furthermore, V is uniformly bounded above by its initial value, because

$$\begin{aligned}V\big(\Delta\theta(t)\big) &= V\big(\Delta\theta(0)\big) + \int_0^t \underbrace{\dot{V}\big(\Delta\theta(s)\big)}_{\leq 0}\, ds\\ &\leq V\big(\Delta\theta(0)\big),\end{aligned}$$

hence $\Delta\theta$ is bounded, from which we conclude that the local estimates $\hat{\theta}_i = \Delta\theta_i + \theta$ and predictions $\hat{y}_i = \hat{\theta}_i\phi(x_i)$ are bounded. This finishes the proof of part 1. Next, we integrate both sides of (10),

$$V(t) - V(0) = -\int_0^t \Delta\theta(\tau)^T L\Delta\theta(\tau) + \gamma\sum_{i=1}^{n}|\Delta y_i(\tau)|^2\, d\tau,$$

and let $t \to \infty$. Note that the prediction errors $\Delta y_i(t)$ are square integrable for all $i = 1, \ldots, n$. Moreover, the quadratic disagreement $\Delta\theta^T L\Delta\theta$ has a finite integral. If we can prove uniform continuity of Δy_i and $\Delta\theta^T L\Delta\theta$, then Barbalat's lemma would imply parts 2 and 3.

The derivative of Δy_i is

$$\frac{d}{dt}\Delta y_i = \Big(\frac{d}{dt}\Delta\theta_i\Big)^T\phi(x_i(t)) + \Delta\theta_i^T D\phi(x_i(t))\dot{x}_i(t), \tag{11}$$

where $D\phi(x_i(t)) \in \mathbf{R}^{p\times q}$ is the Jacobian matrix of ϕ with respect to x evaluated at $x_i(t)$. Since $d\Delta\theta/dt$ is bounded as a result of (9), and $x_i(t)$, $\dot{x}_i(t)$ are bounded by assumption with ϕ continuously differentiable with respect to x, the derivative (11) is bounded. Thus $\Delta y_i \to 0$ as $t \to \infty$, proving part 2. Next,

$$\frac{d}{dt}\Delta\theta^T L\Delta\theta = -\Delta\theta^T(2L^TL + \gamma(\Phi L + L\Phi))\Delta\theta \tag{12}$$

is bounded because it is a sum of bounded terms, thus $\Delta\theta^T L\Delta\theta \to 0$ as $t \to \infty$. In particular, this means $L\Delta\theta = L\hat{\theta} \to 0$, where again we used $L\mathbf{1} = 0$. For a connected graph, the null space of the Laplacian is $\mathrm{null}(L) = \mathrm{span}\{\mathbf{1}\}$, hence $\hat{\theta}_j - \hat{\theta}_i \to 0$ for all $i, j = 1, \ldots, n$. In other words, the parameter estimates asymptotically reach consensus. This completes the proof of part 3.

For part 4, note that the dynamics of $\Delta\theta$ in (9) are linear time-varying, so it suffices to show that the condition in Theorem 2 is met for $P(t) = L + \gamma\Phi(t)$ and $b = 0$.

Let the Laplacian have eigendecomposition $Lv_i = \lambda_i v_i$ with $\lambda_i > 0$ for $i = 2, \ldots, n$. Complete the basis of $\mathbf{R}^n$ so $\{\frac{1}{\sqrt{n}}\mathbf{1}, v_2, \ldots, v_n\}$ is an orthonormal set. Write a unit vector w in this basis as

$$w = \frac{\alpha}{\sqrt{n}}\mathbf{1} + \sum_{j=2}^{n}\beta_j v_j, \tag{13}$$

so that $(\alpha, \beta_2, \ldots, \beta_n) \in \mathbf{R}^n$ has unit norm. Pick $t_0 \geq 0$ and $t > t_0$, and denote the time average of a quantity over the interval $[t_0, t]$ by a bar over the quantity, as in

$$\bar{\Phi} \triangleq \frac{1}{t - t_0}\int_{t_0}^{t}\Phi(\tau)\, d\tau,$$

so that collective PE (4) implies $m_2 \geq \mathbf{1}^T\bar{\Phi}\mathbf{1} \geq m_1$. Then,

$$\begin{aligned}\frac{1}{t-t_0}\int_{t_0}^{t} w^T(L+\gamma\Phi(\tau))w\,d\tau \\ = w^T Lw + w^T\bar{\Phi}w \geq \max\{w^T Lw, \gamma w^T\bar{\Phi}w\}, \quad (14)\end{aligned}$$

because L and $\bar{\Phi}$ are positive semidefinite. Our goal is to bound the maximum (14) away from zero for all time. By substituting (13) into (14) and using $L\mathbf{1} = 0$ and $\mathbf{1}^T L = 0$, we bound the first term in the maximum below by

$$\begin{aligned}w^T Lw &= \sum_{i=2}^{n}\beta_i v_i^T L\sum_{j=2}^{n}\beta_j v_j \\ &= \sum_{i=2}^{n}\sum_{j=2}^{n}\beta_i\beta_j\lambda_j v_i^T v_j = \sum_{i=2}^{n}\lambda_i\beta_i^2 \\ &\geq \lambda_2\|\beta\|_2^2 \\ &= \lambda_2(1-\alpha^2),\end{aligned}$$

where $\beta = (\beta_2, \ldots, \beta_n) \in \mathbf{R}^{n-1}$ and in the last line we used $\|\beta\|_2^2 = 1-\alpha^2$. Next, for $V \triangleq [v_2, \ldots, v_n] \in \mathbf{R}^{n\times n-1}$, the second term has a lower bound

$$\begin{aligned}w^T\bar{\Phi}w &= \frac{\alpha^2}{n}\mathbf{1}^T\bar{\Phi}\mathbf{1} + \underbrace{\beta^T V^T\bar{\Phi}V\beta}_{\geq 0} + \frac{2\alpha}{\sqrt{n}}\mathbf{1}^T\bar{\Phi}V\beta \\ &\geq \frac{\alpha^2}{n}\underbrace{\mathbf{1}^T\bar{\Phi}\mathbf{1}}_{\geq m_1} - \frac{2|\alpha|}{\sqrt{n}}|\mathbf{1}^T\bar{\Phi}V\beta| \\ &\geq \frac{\alpha^2}{n}m_1 - \frac{2|\alpha|}{\sqrt{n}}\underbrace{\|\bar{\Phi}\mathbf{1}\|_1}_{\leq m_2}\underbrace{\|V\beta\|_\infty}_{\leq\|\beta\|_2} \\ &\geq \frac{\alpha^2}{n}m_1 - 2m_2\sqrt{\frac{\alpha^2}{n}(1-\alpha^2)}.\end{aligned}$$

The second line follows form Cauchy-Schwarz and the third from Hölder's inequality. Putting these together gives the required lower bound

$$\max\{w^T Lw, \gamma w^T\bar{\Phi}w\} \geq a > 0,$$

where the worst case rate constant a is

$$\begin{aligned}a = \inf_{|\alpha|\leq 1}\max\Big\{&\lambda_2(1-\alpha^2), \\ &\gamma\frac{\alpha^2}{n}m_1 - 2\gamma m_2\sqrt{\frac{\alpha^2}{n}(1-\alpha^2)}\Big\}. \quad (15)\end{aligned}$$

Note that the infimum in (15) is attained by continuity, and is strictly positive (if a and the first term is zero, then so is the second, $\gamma m_1/n = 0$, a contradiction). ☐

V. Example

Consider the example communication network in Fig. 2 ($n = 3$, $m = 3$, $p = 2$). Three agents are tasked with identifying a true parameter vector $\theta = (\theta_1, \theta_2) = (1, -1) \in \mathbf{R}^2$ using constant regressors. The system to be identified is $y_i(t) = \theta^T\phi_i(t)$. We let $\phi_i : [0, \infty) \to \mathbf{R}^2$ be given by $\phi_i(t) = (c_i, d_i)$, where c_i and d_i are fixed real constants for all $i = 1, 2, 3$.

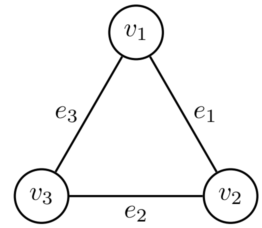


Fig. 2. Communication topology with $n = 3$ agents and $m = 3$ links.

Each ϕ_i is not by itself persistently exciting, as the time average of a constant regressor outer product has rank one:

$$\frac{1}{t-t_0}\int_{t_0}^{t}\begin{bmatrix}c_i\\d_i\end{bmatrix}\begin{bmatrix}c_i & d_i\end{bmatrix}d\tau = \begin{bmatrix}c_i\\d_i\end{bmatrix}\begin{bmatrix}c_i & d_i\end{bmatrix} \not\succeq m_1 I$$

for any $m_1 > 0$, however, the collective PE condition (4) is still satisfied if the ϕ_i are not scalar multiples of the same vector,

$$m_2 I \succeq \frac{1}{t-t_0}\int_{t_0}^{t}\sum_{i=1}^{3}\begin{bmatrix}c_i\\d_i\end{bmatrix}\begin{bmatrix}c_i & d_i\end{bmatrix}d\tau \succeq m_1 I$$

for some $m_1, m_2 > 0$. In other words, collective PE holds for constant regressors provided they span the parameter space $\mathbf{R}^2$. With rate $\gamma = 1$, the parameter estimates $\hat{\theta}_i \in \mathbf{R}^2$ evolve according to

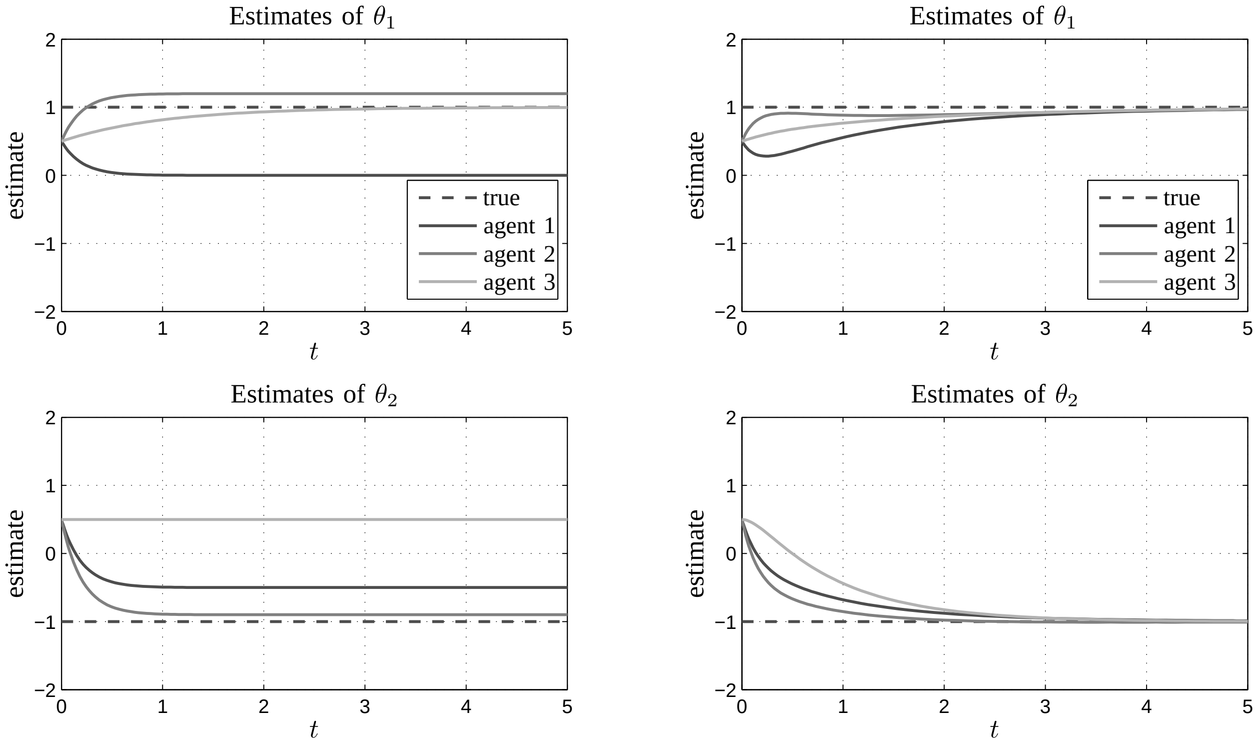
$$\begin{cases}\dot{\hat{\theta}}_1 = -\phi_1(t)(\hat{y}_1 - y_1) + k(\hat{\theta}_2 - \hat{\theta}_1) + k(\hat{\theta}_3 - \hat{\theta}_1)\\ \dot{\hat{\theta}}_2 = -\phi_2(t)(\hat{y}_2 - y_2) + k(\hat{\theta}_3 - \hat{\theta}_2) + k(\hat{\theta}_1 - \hat{\theta}_2)\\ \dot{\hat{\theta}}_3 = -\phi_3(t)(\hat{y}_3 - y_3) + k(\hat{\theta}_1 - \hat{\theta}_3) + k(\hat{\theta}_2 - \hat{\theta}_3).\end{cases}$$

In the estimator dynamics above, consensus terms link the evolution of $\hat{\theta}_i$ to its neighboring $\hat{\theta}_j$ for $j \in \mathcal{N}_i$. Fig. 3 illustrates the parameter estimates as a function of time for each of the three agents with ($k = 1$) and without ($k = 0$) consensus. We used the constant regressors

$$\phi_1(t) = \begin{bmatrix}1\\2\end{bmatrix}, \quad \phi_2(t) = \begin{bmatrix}1\\-2\end{bmatrix}, \quad \phi_3(t) = \begin{bmatrix}1\\0\end{bmatrix}.$$

Without consensus ($k = 0$, Fig. 3a), individual parameter estimates depend solely on underdetermined measurements made at that node, so we have no reason to expect any $\hat{\theta}_i$ to converge to θ. With consensus ($k = 1$, Fig. 3b), the agents collaboratively identify the true parameter.

Isolated agents develop their own (possibly inconsistent) parameter estimates, which replicate their observed input-output relationship. This is indicated in Fig. 4 as a propensity toward the vertical axis. Parameter evolution is frozen once the output prediction error becomes zero, because the local prediction objectives J_i cannot be made any smaller. Collective PE and consensus allow both prediction error and parameter error to approach the origin by adding an extra regularization (disagreement) term to the objective.



(a) Without consensus

(b) With consensus

Fig. 3. (a): individual parameter estimates $\hat{\theta}_i$ fail to converge to θ in a network of three agents without a consensus mechanism in place, because each agent's input is not by itself persistently exciting, (b): with the same inputs and consensus, all parameter estimates to converge to the true value.

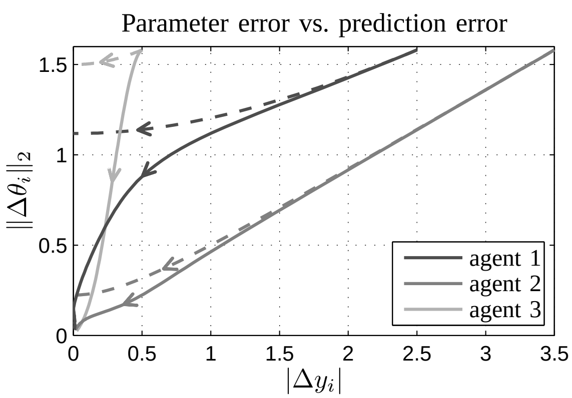


Fig. 4. Prediction error (horizontal axis) tends to zero for all three agents with (solid) and without (dashed) consensus. Parameter error (vertical axis) also tends to zero with consensus due to collective PE.

VI. Conclusion

In this paper we showed that parameter consensus plays an important role in generalizing persistence of excitation to the multi agent setting. Parameter convergence is governed by two main factors: the algebraic connectivity of the communication graph and the level of collective persistence of excitation in the network. Our proof of asymptotic parameter convergence revealed the tension between these two factors, and that a certain kind of ergodicity can allow for parameters to converge even in the absence of leaders in the network that profess their own exciting input. The ideas can be readily extended to model-referenced adaptive control settings, and reinstantiated with all their associated robustness modifications (*e.g.*, σ-mod., projection operations [10], [12]), as well as robustness modifications to consensus itself (*e.g.*, PI control, periodic or sampled updates).

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Appendix

Proof of Theorem 1 for $p > 1$. Form column vectors $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_n) \in \mathbf{R}^{np}$ and $\Delta\theta = (\Delta\theta_1, \ldots, \Delta\theta_n) \in \mathbf{R}^{np}$ by stacking the components $\hat{\theta}_i \in \mathbf{R}^p$ and $\Delta\theta_i = \hat{\theta}_i - \theta \in \mathbf{R}^p$ for all $i = 1, \ldots, n$. The dynamics (9) are now

$$\frac{d}{dt}\Delta\theta = -(L \otimes I_p)\Delta\theta - \gamma\Phi(t)\Delta\theta,$$

where $\otimes$ is the Kronecker product, $I_p \in \mathbf{R}^{p\times p}$ is the identity matrix, and $\Phi : [0, \infty) \to \mathbf{R}^{np\times np}$ is block diagonal,

$$\Phi(t) = \begin{bmatrix} \phi_1(t)\phi_1(t)^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \phi_n(t)\phi_n(t)^T \end{bmatrix}.$$

The candidate Lyapunov function

$$V(\Delta\theta) = \frac{1}{2}\Delta\theta^T\Delta\theta = \frac{1}{2}\sum_{i=1}^{n}\Delta\theta_i^T\Delta\theta_i$$

has nonpositive derivative

$$\dot{V}(\Delta\theta) = -\Delta\theta^T\big((L\otimes I_p) + \gamma\Phi(t)\big)\Delta\theta \le 0,$$

with $\dot{V} \to 0$ as $t \to \infty$ by the same arguments as before, thus parts 1-3 follow. For part 4, the mixed product property $AB\otimes CD = (A\otimes C)(B\otimes D)$ for appropriately sized matrices A, B, C, and D implies that the spectrum of $L\otimes I_p$ is related to the spectrum of L and I_p by

$$(L\otimes I_p)\left(\frac{1}{\sqrt{n}}\mathbf{1}\otimes e_j\right) = 0,$$
$$(L\otimes I_p)\,(v_i\otimes e_j) = \lambda_i\,(v_i\otimes e_j)\,,$$

for all $i = 2,\ldots,n$ and $j = 1,\ldots,p$, where $e_j \in \mathbf{R}^p$ is the jth unit vector. Write a unit vector $w \in \mathbf{R}^{np}$ in this basis as

$$w = \sum_{j=1}^{p}\alpha_j\frac{1}{\sqrt{n}}\mathbf{1}\otimes e_j + \sum_{i=2}^{n}\sum_{j=1}^{p}\beta_{ij}v_i\otimes e_j,$$

with $(\alpha,\beta) \in \mathbf{R}^p \times \mathbf{R}^{(n-1)p}$ having unit norm. As before, let $\bar{\Phi}$ be the average of Φ over $[t_0,t]$. We wish to bound

$$\max\{w^T(L\otimes I_p)w, \gamma w^T\bar{\Phi}w\}$$

uniformly below by a strictly positive constant. Using the mixed product property and $\|\alpha\|_2^2 + \|\beta\|_2^2 = 1$ we have

$$\begin{aligned} w^T(L\otimes I_p)w &= \sum_{i=2}^{n}\sum_{j=1}^{p}\lambda_i\beta_{ij}^2 \\ &\ge \lambda_2(1-\|\alpha\|_2^2). \end{aligned}$$

For the second term $w^T\bar{\Phi}w$, note that

$$(\mathbf{1}\otimes e_i)^T\bar{\Phi}(\mathbf{1}\otimes e_j) = (\overline{\phi_1\phi_1^T})_{ij} + \cdots + (\overline{\phi_n\phi_n^T})_{ij},$$

hence

$$\begin{aligned} w^T\bar{\Phi}w = {}& \frac{1}{n}\sum_{i=1}^{p}\sum_{j=1}^{p}\alpha_i\alpha_j(\mathbf{1}\otimes e_i)\bar{\Phi}(\mathbf{1}\otimes e_j) \\ &+ \frac{2}{\sqrt{n}}\sum_{i=2}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p}\alpha_k\beta_{ij}(\mathbf{1}\otimes e_k)^T\bar{\Phi}(v_i\otimes e_j) \\ &+ \underbrace{\sum_{i=2}^{n}\sum_{j=1}^{p}\sum_{k=2}^{n}\sum_{l=1}^{p}\beta_{ij}\beta_{kl}(v_i\otimes e_j)\bar{\Phi}(v_k\otimes e_l)}_{\ge 0} \\ \ge {}& \frac{1}{n}\alpha^T\left(\sum_{i=1}^{n}\overline{\phi_i\phi_i^T}\right)\alpha \\ &- \frac{2m_2n}{\sqrt{n}}\sum_{i=2}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p}|\alpha_k\beta_{ij}| \\ \ge {}& \frac{\|\alpha\|_2^2}{n}m_1 - 2m_2n\sqrt{\|\alpha\|_2^2(1-\|\alpha\|_2^2)}, \end{aligned}$$

thus a loose uniform lower bound is

$$\max\{w^T(L\otimes I_p)w, \gamma w^T\bar{\Phi}w\} \ge a > 0.$$

A continuity argument should convince the reader that

$$\begin{aligned} a = \inf_{\|\alpha\|_2\le 1}\max\Big\{&\lambda_2(1-\|\alpha\|_2^2), \\ &\gamma\frac{\|\alpha\|_2^2}{n}m_1 - 2\gamma m_2n\sqrt{\|\alpha\|_2^2(1-\|\alpha\|_2^2)}\Big\} \end{aligned}$$

is strictly positive. $\square$

References

[1] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[2] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[3] M. Schwager, D. Rus, and J.-J. Slotine, "Decentralized, adaptive coverage control for networked robots," *International Journal of Robotics Research*, vol. 28, no. 3, pp. 357–375, Mar. 2009.

[4] W. Wang and J.-J. Slotine, "A theoretical study of different leader roles in networks," *IEEE Transactions on Automatic Control*, vol. 51, no. 7, pp. 1156–1161, 2006.

[5] S. S. Stanković, M. S. Stanković, and D. M. Stipanović, "Decentralized parameter estimation by consensus based stochastic approximation," *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 531–543, 2011.

[6] M. Guo and D. V. Dimarogonas, "Nonlinear consensus via continuous, sampled, and aperiodic updates," *International Journal of Control*, vol. 86, no. 4, pp. 567–578, 2013.

[7] A. P. Morgan and K. S. Narendra, "On the uniform asymptotic stability of certain linear nonautonomous differential equations," *SIAM Journal on Control and Optimization*, vol. 15, no. 1, pp. 5–24, 1977.

[8] B. D. O. Anderson, "Exponential stability of linear equations arising in adaptive identification," *IEEE Transactions on Automatic Control*, vol. 22, no. 1, pp. 83–88, 1977.

[9] S. P. Boyd and S. Sastry, "On parameter convergence in adaptive control," *Systems and Control Letters*, vol. 3, no. 6, pp. 311–319, Dec. 1983.

[10] P. Ioannou and B. Fedan, *Adaptive Control Tutorial*, ser. Advances in Design and Control. SIAM, 2006.

[11] S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence and Robustness*. Prentice Hall, 1989.

[12] E. Lavretsky and K. A. Wise, *Robust and Adaptive Control: with Aerospace Applications*, ser. Advanced Textbooks in Control and Signal Processing. Springer, 2013.

[13] W. Yu, G. Chen, M. Cao, and J. Kurths, "Second-order consensus for multiagent systems with directed topologies and nonlinear dynamics," *IEEE Transactions on Systems, Man, and Cybernetics—Part B: Cybernetics*, vol. 40, no. 3, pp. 881–891, June 2010.

[14] W. Ren, "On consensus algorithms for double-integrator dynamics," *IEEE Transactions on Automatic Control*, vol. 53, no. 6, pp. 1503–1509, 2008.

[15] J. S.-C. Yuan and W. M. Wonham, "Probing signals for model reference identification," *IEEE Transactions on Automatic Control*, vol. 22, no. 4, pp. 530–538, 1977.

[16] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[17] C. Godsil and G. Royle, *Algebraic Graph Theory*, ser. Graduate Texts in Mathematics. Springer, 2001, vol. 207.